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On the numerical solution of a driven thin film equation

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Abstract

This paper is devoted to comparing numerical schemes for a differential equation with convection and fourth-order diffusion. Our model equation is $u_t + (u^2 - u^3)_x = -(u^3 u_{xxx})_x$, which arises in the context of thin film flow. First we employ implicit schemes and treat both convection and diffusion terms implicitly. Then the convection terms are treated with well-known explicit schemes, namely Godunov, WENO and an upwind-type scheme, while the diffusion term is still treated implicitly. The diffusion and convection schemes are combined using a fractional step-splitting method.

1. Introduction

In this paper we consider numerical solutions to the following equation

$$u_t + f(u)_x = -(u^3 u_{xxx})_x, \tag{1.1}$$

where the flux is given by

$$f(u) = u^2 - u^3. \tag{1.2}$$

Eq. (1.1) describes the flow of a thin liquid film, where $u(x,t) \geqslant 0$ denotes the film thickness. The flux terms represent surface shear and gravity, where the forces act in opposing directions, the diffusion term on the right hand side represents surface tension. The surface shear term may arise due to temperature or concentration gradients or to an external shear force (caused by wind for example). Derivations of Eq. (1.1) and related equations may be found in the reviews [21,25]. For the specific case when thermocapillary effects produce the surface shear, Eq. (1.1) is derived in [4,11], with a wind induced stress a derivation is given in [22,23]. Experimental results showing typical film shapes for thermocapillary flow up a vertical plate are presented in [11]. The experiments show good agreement with numerical solutions for the small times that the

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experiments have been run [3]. However, it is the properties of the numerical solution that we are focussing on, not the comparison with experiment.

The numerical solution of Eqs. (1.1) and (1.2) is constrained by the diffusion term. An explicit scheme requires a time-step Δt of the order $(\Delta x)^4$. Consequently in regions where high resolution is required, such as at a moving front, a singularity or at blow-up, the computational time is prohibitive. Implicit methods are therefore generally preferred. Recently these have been coupled with adaptive meshes to permit high accuracy in the regions of primary interest, see [2,12,27,33] for example. However, the first-order convection term is not subject to the same constraint and there are many different methods to deal with nonlinear convection. In the following work we focus primarily on a comparison between finite difference, Godunov, an adapted upwind and WENO schemes applied to the convection term. We also investigate the effect of applying fully implicit and Crank–Nicolson schemes. Even if it is possible to solve the full equation in a single step with implicit schemes, fractional step splitting, alternating between solving for the diffusion and convection terms, is applied in all cases for consistency in the tests.

The majority of our numerical examples will be taken from [4]. We use these examples because Bertozzi et al. [4] present a very careful numerical and analytical investigation of Eqs. (1.1) and (1.2) and the cases presented show a wide variety of behaviour in the solutions. The flux function has a point of inflexion at $u = 1/3$. The form of solution is likely to change around this point, consequently in our numerical solutions we will take limiting values for u close to this value.

2. Numerical schemes

The notation employed in the numerical calculations is as follows. We consider a uniform mesh $x_{j+1/2}$ with a fixed width $h \equiv \Delta x > 0$, where $x_{j+1/2} = (j + 1/2)h$, $j \in \mathbf{Z}$. The time mesh is given by $t^n = n\Delta t$, with a fixed time step size $\Delta t > 0$. A solution to a nonlinear convection equation may have a discontinuity and its numerical correspondence U_j^n is usually considered as the approximation to the cell average of the true solution, i.e.,

$$U_j^n \cong \frac{1}{h}\int_{x_{j-1/2}}^{x_{j+1/2}} u(x, t^n)\mathrm{d}x. \tag{2.1}$$

We use a fractional step-splitting method to handle two terms in Eq. (1.1). The convection term will be tackled via various implicit and explicit finite difference (or finite volume) methods which will be given in the following section. We will observe the performance differences made by these schemes for the convection part. The diffusion term will always be dealt with via an implicit method.

2.1. Finite difference for the diffusion term

First consider a finite difference scheme for the diffusion equation

$$u_t = -\phi(u)_x, \quad \phi(u) = u^3 u_{xxx}. \tag{2.2}$$

We view $u_{j+1/2}^n$ as a time average of $u(x,t)$ on the interval $t \in [t^n, t^{n+1}]$ at the interface $x = x_{j+1/2}$. Then, after integrating (2.2) over the mesh $[x_{j-1/2}, x_{j+1/2}] \times [t^n, t^{n+1}]$, one can easily check that the cell averages given by (2.1) satisfy

$$U_j^{n+1} = U_j^n - \frac{\Delta t}{h}(\phi(u_{j+1/2}^n) - \phi(u_{j-1/2}^n)), \tag{2.3}$$

where we view $\phi(u_{j+1/2}^n)$ as a time average of the diffusive flux on the interval $t \in [t^n, t^{n+1}]$ at the interface $x = x_{j+1/2}$. Since

$$u(x + 2h) - 3u(x + h) + 3u(x) - u(x - h) = h^3 u_{xxx}(x + h/2) + \mathrm{O}(h^5),$$

we obtain the following finite difference representation

$$h^3\phi(u_{j+1/2}^n) \cong \left(\frac{U_{j+1}^n + U_j^n}{2}\right)^3 (U_{j+2}^n - 3U_{j+1}^n + 3U_j^n - U_{j-1}^n) =: \Phi_{j+1/2}(U^n).$$