Available online at www.sciencedirect.com

ScienceDirect

Journal of Computational Physics 227 (2008) 7294–7321

JOURNAL OF COMPUTATIONAL PHYSICS

www.elsevier.com/locate/jcp

Development of nonlinear weighted compact schemes with increasingly higher order accuracy

Shuhai Zhang [a,1], Shufen Jiang [a], Chi-Wang Shu [b,*,2]

[a] China Aerodynamics Research and Development Center, Mianyang, Sichuan 621000, China
[b] Division of Applied Mathematics, Brown University, Providence, RI 02912, USA

Received 24 December 2007; received in revised form 12 April 2008; accepted 14 April 2008
Available online 25 April 2008

Abstract

In this paper, we design a class of high order accurate nonlinear weighted compact schemes that are higher order extensions of the nonlinear weighted compact schemes proposed by Deng and Zhang [X. Deng, H. Zhang, Developing high-order weighted compact nonlinear schemes, J. Comput. Phys. 165 (2000) 22–44] and the weighted essentially non-oscillatory schemes of Jiang and Shu [G.-S. Jiang, C.-W. Shu, Efficient implementation of weighted ENO schemes, J. Comput. Phys. 126 (1996) 202–228] and Balsara and Shu [D.S. Balsara, C.-W. Shu, Monotonicity preserving weighted essentially non-oscillatory schemes with increasingly high order of accuracy, J. Comput. Phys. 160 (2000) 405–452]. These nonlinear weighted compact schemes are proposed based on the cell-centered compact scheme of Lele [S.K. Lele, Compact finite difference schemes with spectral-like resolution, J. Comput. Phys. 103 (1992) 16–42]. Instead of performing the nonlinear interpolation on the conservative variables as in Deng and Zhang (2000), we propose to directly interpolate the flux on its stencil. Using the Lax–Friedrichs flux splitting and characteristic-wise projection, the resulted interpolation formulae are similar to those of the regular WENO schemes. Hence, the detailed analysis and even many pieces of the code can be directly copied from those of the regular WENO schemes. Through systematic test and comparison with the regular WENO schemes, we observe that the nonlinear weighted compact schemes have the same ability to capture strong discontinuities, while the resolution of short waves is improved and numerical dissipation is reduced.

Keywords: Compact scheme; Weighted interpolation; WENO scheme; Smoothness indicator

1. Introduction

There are two typical approaches to design high order finite difference schemes for solving partial differential equations. The first is the traditional concept that the derivative of a function on the numerical grid is

* Corresponding author. Tel.: +1 401 863 2549; fax: +1 401 863 1355.
E-mail addresses: zhang_shuhai@tom.com (S. Zhang), shu@dam.brown.edu (C.-W. Shu).
[1] Research supported by the Chinese National Natural Science Foundation Grants 10572146 and 10772193.
[2] Research partially supported by NSF Grants DMS-0510345 and AST-0506734.

0021-9991/$ - see front matter
doi:10.1016/j.jcp.2008.04.012

approximated by a linear combination of the function on a subset of the grid (stencil). The linear combination coefficients should satisfy certain order conditions in order to achieve a high order accurate approximation to the derivative. This is the standard finite difference method that is called a non-compact finite difference scheme by Adams and Shariff [1]. The second approach to design finite difference schemes, corresponding to the so-called compact schemes, is to form a linear combination of the unknown approximations to the derivative at the grid points in a stencil, and equate it with another linear combination of the function itself at the grid points in the same stencil. The word "compact" corresponds to the fact that for the same order of accuracy, the stencil can be more compact in the second approach. However, a linear system must be solved to obtain approximations to the derivative at the grid points for compact schemes, thus the effective stencil for the approximation of the derivative at a grid point, in terms of the function values in the mesh, is not compact at all. The most influential reference for compact schemes is [26].

The weighted essentially non-oscillatory (WENO) finite difference scheme [19] is a typical high order non-compact finite difference scheme suitable for solving convection dominated partial differential equations containing possible discontinuities in the solutions, such as the Euler or Navier–Stokes equations in computational fluid dynamics. It is an extension of the essentially non-oscillatory (ENO) scheme which was introduced by Harten et al. [15]. The accuracy can be improved to the optimal order in smooth regions while the essentially non-oscillatory property near discontinuities is maintained. The WENO idea was first introduced by Liu et al. [27], in which the authors used a cell average approach (finite volume framework) to convert an rth order ENO scheme to an $(r+1)$th order WENO scheme. Based on the pointwise finite difference ENO scheme [38,39] and by a careful design of the smoothness indicator and nonlinear weights, the WENO scheme in [19] can achieve the optimal $(2r-1)$th order accuracy when converting an rth order ENO scheme, while still keeping the essentially non-oscillatory property near shock waves. The WENO schemes have the two desirable properties that they capture discontinuities and maintain high order accuracy. It has been applied to many problems containing discontinuous solutions. We refer to the recent review paper [37] for more details.

Even though the order of accuracy for explicit finite difference WENO schemes can be designed to be arbitrarily high, such as the eleventh order WENO scheme developed by Balsara and Shu [2], the resolution of short waves of such high order explicit finite difference schemes is not ideal. The order of accuracy refers to the asymptotic behavior of the scheme for solving smooth solutions when the mesh size becomes small. In applications, for example in wave dominated problems such as aeroacoustics and turbulence, we often need to approximate solutions on a relatively coarse mesh with respect to the wave frequencies that we would like to resolve. The scheme's ability to resolve short wavelengths relative to a given mesh can be represented by a dispersion relation. The best method to simulate wave dominated problems is the spectral method [4,11,22], which is high order accurate and has the best dispersion relation. However, the spectral method has its own limitation as it imposes significant restrictions on the geometry and boundary conditions. Typical explicit high order finite difference schemes, corresponding to the choice of linear combination coefficients to maximize the order of accuracy for a given stencil, do not have optimal dispersion relations. To overcome this drawback, there are efforts in the literature to modify the linear combination coefficients in a finite difference scheme to improve its dispersion relation, at the price of lowering the achievable order of accuracy corresponding to a given stencil. Tam and Webb [42] used this strategy to develop a dispersion relation preserving (DRP) finite difference scheme. Ponziani et al. [33] and Wang and Chen [43] also used this strategy to develop optimal WENO schemes for dispersion relationships.

A good choice to simulate wave dominated problems is the compact scheme, which typically has better dispersion relation than a finite difference scheme of the same order of accuracy. Early discussion of compact schemes can be found in [17,23]. In [26], Lele developed a family of compact schemes for the first and second derivatives. Through systematic Fourier analysis, it is shown that these compact schemes have spectral-like resolution for short waves. In practice, compact approximations on a cell-centered mesh has superiority due to their smaller numerical viscosity. Nagarajan et. al [31] and Boersma [3] used staggered mesh compact schemes to simulate compressible flows. Numerical tests indicate that their methods are quite robust. Through coupling the second derivatives, Mahesh [28] developed a family of compact schemes with good spectral-like resolution. Shukla and Zhong [40] developed a compact scheme for non-uniform meshes. Upwind compact schemes were also developed [6,10,48] for solving nonlinear hyperbolic problems. The compact schemes have