Journal of Computational Physics 230 (2011) 8665–8668

Contents lists available at SciVerse ScienceDirect

Journal of Computational Physics

journal homepage: www.elsevier.com/locate/jcp

Short Note

On the treatment of contact discontinuities using WENO schemes

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ARTICLE INFO

Article history:
Received 20 June 2011
Received in revised form 21 July 2011
Accepted 18 August 2011
Available online 5 September 2011

Keywords:
Shock capturing
WENO
Contact discontinuities

1. Introduction

High-order accurate shock-capturing methods based on weighted essentially non-oscillatory schemes (WENO, [3]) have been used extensively to simulate compressible flows with shocks. Generally, the WENO reconstruction is performed directly on the conservative variables (component-wise) or in characteristic space. For shock-tube problems, the latter usually produces solutions with fewer oscillations. However, because of the computational cost or difficulties in calculating the transformation matrices, component-wise reconstruction is sometimes preferred.

It is shown in this work that small numerical errors (up to 0.1%) are generated when computing isolated *contact discontinuities* in a fluid of uniform composition using a component-wise WENO reconstruction. Although small, such errors may be of the order of physical disturbances, *e.g.*, in turbulence or acoustics, and thus overwhelm them. These errors, illustrated in Fig. 1, are similar to those generated in simulations of interfaces separating fluids of different specific heats ratios [1], though approximately an order of magnitude smaller. As shown in this work, these errors may be prevented either by reconstructing the primitive variables [2] or using a characteristic reconstruction.

An analysis inspired by that of [1] for material interfaces is applied to contact discontinuities. The one-dimensional Euler equations are considered:

$$q_t + [f(q)]_x = 0, \quad q = (\rho, \rho u, E)^T, \quad f = (\rho u, \rho u^2 + p, (E + p)u)^T, \tag{1}$$

where ρ is the density, u is the velocity, E is the total energy, $p = (\gamma - 1)(E - \rho u^2/2)$ is the pressure and $\gamma = 1.4$ is the ratio of specific heats, *e.g.*, of air.

Across an isolated contact discontinuity, the velocity and pressure are constant, but the temperature T, and therefore the density, are discontinuous. Initial conditions corresponding to Fig. 1 are

$$T = \begin{cases} 1/2, & \text{if } -0.5 \leqslant x \leqslant 0.5, \\ 1, & \text{otherwise}, \end{cases} \quad \rho = 1/T, \quad u = 1, \quad p = 1/\gamma. \tag{2}$$

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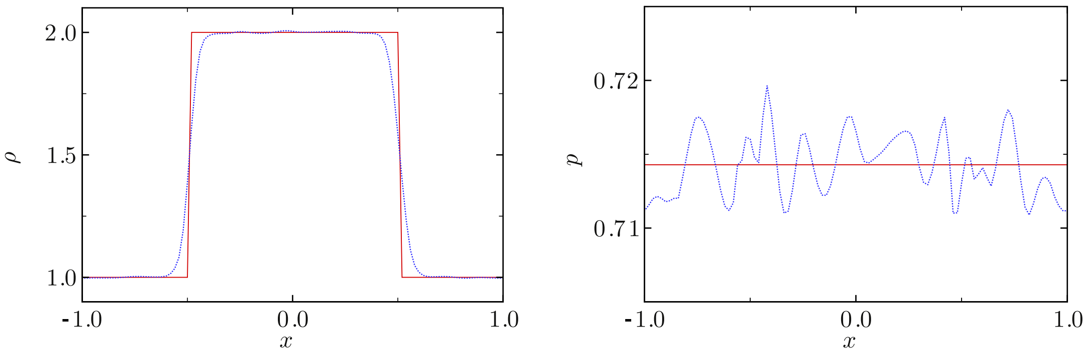
0021-9991/$ - see front matter
doi:10.1016/j.jcp.2011.08.017



(a) Density (b) Pressure

Fig. 1. Advection of a contact discontinuity using a component-wise WENO reconstruction of the conservative variables. Solid red line: initial condition; dotted blue line: solution after one period. (For interpretation of the references in colour in this figure legend, the reader is referred to the web version of this article.)

If the contact discontinuity is located in cell i at time t^n, the mass, momentum and energy conservation equations can be marched forward by one time step or substep:

$$\rho_i^{n+1} = \rho_i^n - \frac{\Delta t}{\Delta x}\mathcal{D}_i^{\rho}(\rho u), \tag{3a}$$

$$(\rho u)_i^{n+1} = (\rho u)_i^n - \frac{\Delta t}{\Delta x}\left[\mathcal{D}_i^{\rho u}(\rho u u) + \mathcal{D}_i(p)\right], \tag{3b}$$

$$E_i^{n+1} = E_i^n - \frac{\Delta t}{\Delta x}\left[\mathcal{D}_i^{E}(\rho e u) + \mathcal{D}_i^{E}\left(\rho\frac{u^2}{2}u\right) + \mathcal{D}_i(up)\right]. \tag{3c}$$

In the continuous case, Eqs. (3) all reduce to the same advection equation for density and the velocity and pressure remain constant for all time. Achieving this behavior is not trivial for the discrete equations, as exhibited in Fig. 1.

A finite volume formulation is used with the Lax–Friedrichs solver, in which case the difference operators are

$$\mathcal{D}_i^{a}(au) = \frac{a_{i+1/2}^R u_{i+1/2}^R + a_{i+1/2}^L u_{i+1/2}^L}{2} - \frac{a_{i-1/2}^R u_{i-1/2}^R + a_{i-1/2}^L u_{i-1/2}^L}{2} - \frac{\alpha}{2}\left(a_{i+1/2}^R - a_{i+1/2}^L - a_{i-1/2}^R + a_{i-1/2}^L\right), \tag{4}$$

where the superscript of $\mathcal{D}_i^a$ denotes the variable a on which the WENO weights are based, L and R are the left and right reconstructed states and α is the largest eigenvalue in the domain, and

$$\mathcal{D}_i(a) = \frac{a_{i+1/2}^R + a_{i+1/2}^L}{2} - \frac{a_{i-1/2}^R + a_{i-1/2}^L}{2}, \tag{5}$$

for a given property a. Without loss of generality, the left cell edge values are considered in a third-order accurate WENO reconstruction. The analysis can readily be extended to higher order and other solvers, combinations of left and right states, and finite difference.

The velocity, $u_i = \rho u_i/\rho_i = u$, and pressure, $p_i = (\gamma - 1)[E_i - \rho u_i)^2/2\rho_i] = p$, are constant at time t^n. From Eqs. (3), the velocity and pressure at time t^{n+1} are given by

$$u_i^{n+1} = u\frac{\rho_i^n - \frac{\Delta t}{\Delta x}\mathcal{D}_i^{\rho u}(\rho u)}{\rho_i^n - \frac{\Delta t}{\Delta x}\mathcal{D}_i^{\rho}(\rho u)} - \frac{\frac{\Delta t}{\Delta x}\mathcal{D}_i(p)}{\rho_i^n - \frac{\Delta t}{\Delta x}\mathcal{D}_i^{\rho}(\rho u)}, \tag{6}$$

$$p_i^{n+1} = (\gamma - 1)\left[\frac{p_i^n}{(\gamma - 1)} - \frac{\Delta t}{\Delta x}\mathcal{D}_i^{E}\left(\frac{p}{\gamma - 1}u\right) - \frac{\Delta t}{\Delta x}\mathcal{D}_i(up) + \frac{\rho u_i^2}{2} - \frac{\Delta t}{\Delta x}\mathcal{D}_i^{E}\left(\rho\frac{u^2}{2}u\right) - \frac{1}{2}\frac{\left\{\rho u_i^n - \frac{\Delta t}{\Delta x}\left[\mathcal{D}_i^{\rho u}(\rho u u) + \mathcal{D}_i(p)\right]\right\}^2}{\rho_i^n - \frac{\Delta t}{\Delta x}\mathcal{D}_i^{\rho}(\rho u)}\right]. \tag{7}$$

It is shown below that density is reconstructed inconsistently in the three equations and that, as a result, an error is introduced in the velocity and pressure.

The density, momentum and energy reconstructed using WENO at the cell edges are: