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On the parametric finite element approximation of evolving hypersurfaces in $\mathbb{R}^3$

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Abstract

We present a variational formulation of motion by minus the Laplacian of curvature and mean curvature flow, as well as related second and fourth order flows of a closed hypersurface in $\mathbb{R}^3$. On introducing a parametric finite element approximation, we prove stability bounds and compare our scheme with existing approaches. The presented scheme has very good properties with respect to the distribution of mesh points and, if applicable, volume conservation.

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1. Introduction

In this paper we analyze a parametric finite element approximation for the evolution of closed hypersurfaces $\Gamma \subset \mathbb{R}^d$, $d = 3$, moving under given geometric flows such as motion by mean curvature and motion by surface diffusion. The present authors introduced the scheme considered here in [7] for fourth order geometric evolution equations and extended it in various ways, including to the case of second order equations and the presence of external boundaries, in [5]. In both of these papers, only curves and networks of curves in the plane ($d = 2$) were considered. Here we recall, that in this case an intrinsic discrete tangential motion induced by our scheme leads to an almost uniform distribution of nodes along the polygonal approximation of the curve Γ. One aim of this paper will be, to investigate whether this remarkable property extends to surfaces in $\mathbb{R}^3$.

Our approach makes use of a fundamental idea of Dziuk, see [12], who used the identity

$$\Delta_s \vec{x} = \vec{\varkappa} \equiv \varkappa \vec{\nu} \tag{1.1}$$

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for the first time in order to design a finite element method for geometric partial differential equations and mean curvature flow; see also [13]. The identity (1.1) is well known from surface geometry, where Δ_s is the surface Laplacian (Laplace–Beltrami operator), $\vec{x}$ is a parameterization of Γ, $\vec{\varkappa}$ is the mean curvature vector with $\varkappa$ the sum of the principal curvatures and $\vec{\nu}$ a unit normal to Γ. Here one uses the sign convention that $\varkappa$ is positive if the surface is curved in the direction of the normal. A second idea stems from [4], where a splitting method and a solver based on a Schur complement approach were proposed in order to compute solutions of the surface diffusion law

$$\mathcal{V} = -\Delta_s \varkappa, \tag{1.2}$$

where $\mathcal{V}$ is the normal velocity of the surface.

The motion of surfaces driven by second or fourth order geometric evolution equations arises in many applications in materials science and in differential geometry. For a closed hypersurface Γ in $\mathbb{R}^d$, which evolves in time, motion by surface diffusion is given by (1.2). The mean curvature flow, on the other hand, is a second order evolution equation and is given by

$$\mathcal{V} = \varkappa. \tag{1.3}$$

In this paper, we are also going to consider more general flows of the form

$$\mathcal{V} = f(\varkappa), \tag{1.4}$$

where $f : (a, b) \to \mathbb{R}$ with $-\infty \leqslant a < b \leqslant \infty$, is a strictly monotonically increasing continuous function, e.g.

$$f(r) := |r|^{\beta-1} r, \quad \beta \in \mathbb{R}_{>0}, \tag{1.5}$$

see [21] and the references therein. For example, in the curve case ($d = 2$), the evolution law (1.4), with (1.5) for $\beta = \frac{1}{3}$, has been studied in [1,25,2]. Of particular interest is the choice

$$f(r) := -r^{-1}, \tag{1.6}$$

i.e. the inverse mean curvature flow, see e.g. [17,20] for the origins of this flow in mathematical physics, where it occurs in the context of the positive mass conjecture; and [19], and the references therein, for a consideration of this flow in differential geometry. Numerical results for the inverse mean curvature flow of surfaces in $\mathbb{R}^3$ have been given in [23], where a finite volume approximation of a regularized level-set formulation of (1.4) with (1.6) is considered. For $d = 3$ we know of no other approach for the approximation of the inverse mean curvature flow in the literature. For parameterizations $\vec{x} : \Omega \times [0, T] \to \mathbb{R}^d$ of Γ, where Ω is a suitable compact reference manifold without boundary in $\mathbb{R}^d$, (1.4) can be written as a second order equation:

$$\mathcal{V} := \vec{x}_t \cdot \vec{\nu} = f(\varkappa), \quad \varkappa\vec{\nu} = \Delta_s \vec{x}. \tag{1.7}$$

Note that because the tangential component $\vec{x}_t - (\vec{x}_t \cdot \vec{\nu})\vec{\nu}$ of the velocity $\vec{x}_t$ is not prescribed in (1.7), there exists a whole family of solutions $\vec{x}$, even though the evolution of Γ is uniquely determined.

A version of (1.4) that preserves the enclosed volume is given by

$$\mathcal{V} = f(\varkappa) - \frac{\int_\Gamma f(\varkappa)\,\mathrm{d}s}{\int_\Gamma 1\,\mathrm{d}s}, \tag{1.8}$$

the so called conserved mean curvature flow, also called surface attachment limited kinetics (SALK), if $f(r) := r$. An intermediate law between (1.4), with $f(r) := r$, and (1.2) is the following evolution law

$$\mathcal{V} = -\Delta_s \left(\frac{1}{\alpha} - \frac{1}{\xi} \Delta_s \right)^{-1} \varkappa, \tag{1.9}$$

where $\alpha, \xi \in \mathbb{R}_{>0}$. The flow (1.9) interpolates between surfaces diffusion (1.2) and SALK, (1.8) with $f(r) := r$, and was first discussed in [28]; see also [14]. It is similar to (1.2) and (1.8) in that the enclosed volume is conserved while the area of the hypersurface decreases. We observe that for $\alpha \to \infty$ and $\xi = 1$, the solutions to (1.9) should converge to solutions of (1.8) with $f(r) := r$, while $\xi \to \infty$ and $\alpha = 1$ corresponds to the law (1.2). The former limit has been rigorously shown in the curve case ($d = 2$), see [15]. Given parameterizations $\vec{x} : \Omega \times [0, T] \to \mathbb{R}^d$ of Γ, (1.9) can be written as a system of second order equations: