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Crank–Nicolson/quasi-wavelets method for solving fourth order partial integro-differential equation with a weakly singular kernel ☆

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A B S T R A C T

In this paper, we study a novel numerical scheme for the fourth order partial integro-differential equation with a weakly singular kernel. In the time direction, a Crank–Nicolson time-stepping is used to approximate the differential term and the product trapezoidal method is employed to treat the integral term, and the quasi-wavelets numerical method for space discretization. Our interest in the present paper is a continuation of the investigation in Yang et al. [33], where we study discretization in time by using the forward Euler scheme. The comparisons of present results with the previous ones show that the present scheme is more stable and efficient for numerically solving the fourth order partial integro-differential equation with a weakly singular kernel. We also tested the method proposed on several one and two dimensional problems with very promising results. Besides, in order to demonstrate the power of the quasi-wavelets method in comparison with standard discretization methods we also consider the high-frequency oscillation problems with the integro-differential term.

1. Introduction

Most recently, fourth-order problems have been of great interest. It is caused both by the intensive development of the theory of fourth order differential equation itself (such as [13]) and by broad applications of such the modeling of thin beams and plates, strain gradient elasticity, and phase separation in binary mixtures, which are basic elements in engineering structures and are of great practical significance to civil, mechanical, and aerospace engineering. Furthermore, the fourth-order problems are omnipresent in modern science and engineering. For example, bridge slabs, floor systems, window glasses, and airplane wings can be modeled as plates with various boundary supports which are governed by fourth-order differential equations.

The objective of our present work is to introduce the quasi-wavelets scheme for the following fourth order partial integro-differential equation with a weakly singular kernel

$$\frac{\partial u(t,x)}{\partial t} + \int_0^t (t-s)^{-\alpha} Au(s,x)ds = f(t,x), \quad x \in \Omega, \quad t > 0, \quad 0 < \alpha < 1, \tag{1}$$

subject to the initial conditions

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$$u(0,x) = \psi(x), \quad x \in \Omega, \tag{2}$$

where $\Omega = [0,1]^d$, A is a fourth-order self-adjoint positive-definite partial differential operator. For the sake of simplification of the paper, we do not iterate three types boundary conditions discussed in our previous paper [33]. In this paper, we only consider the clamped boundary conditions

$$\begin{cases} u(t,x) = 0, & t > 0, \quad x \in \partial\Omega, \\ \frac{\partial u}{\partial n}(t,x) = 0, & t > 0, \quad x \in \partial\Omega, \end{cases} \tag{3}$$

where $\partial\Omega$ is the boundary of Ω, and $\frac{\partial}{\partial n}$ is the outward normal derivative on the boundary $\partial\Omega$. A number of elegant numerical methodology have been introduced for the fourth-order problems in the past few decades. These success include finite difference scheme [6,7], finite element methods [4,11], the homotopy perturbation [8], spectral-Galerkin method [21,22], spectral element methods [34], domain decomposition method [10], finite volume method [17]. And in recent years, Mehrdad and Mehdi [16] developed chebyshev cardinal function method for fourth-order integro-differential equations, but the equation does not include the terms of partial derivative of time and the weakly singular kernel. The local discontinuous Galerkin (LDG) method was introduced by Dong and Shu in [4] for fourth-order time-dependent equation, where they rewrite the fourth-order equation into a system of first-order equations, and then apply DG method to the first-order system. To the best of the author's knowledge, however, no numerical method has been derived so far for solving Eqs. (1)–(3) (see [5,9,15,32] for theory and applications). As pointed out by Li and Xu in [12], the general form of Eq. (1) can be reformulated into fractional partial differential equation (FPDE). Because of the integral in the definition of the noninteger order derivatives, it is apparent that these derivatives are nonlocal operators and possess a memory effect. These feature of the fractional derivatives make the design of accurate and fast methods difficult. Therefore, finding numerical solutions for Eqs. (1)–(3) is still a challenge. However, in this paper we propose quasi-wavelets scheme for the fourth order partial integro-differential equations to overcome those drawbacks. It is the first time and the first person that such a class of problems have been tackled with quasi-wavelets scheme.

The quasi-wavelets method has emerged as an intriguing alternative in many situations–and as a superior one in a few cases for solving partial differential equations [28,36]. The most distinguishing feature of the quasi-wavelets algorithm is its high level of accuracy and reliability. At present, it is the only available method that is able to accurately predict thousands of vibration modes of plates and beams without encountering numerical instability [29,37]. It has been shown that in the framework of the quasi-wavelets algorithm, the quasi-wavelets method is equivalent to a wavelet Galerkin [27]. By appropriately selecting parameters of a quasi-wavelet kernel, the quasi-wavelets approach exhibits spectral accuracy for integration [1,2] and shows great flexibility in handling complex geometries and boundary conditions [20,30]. The theory of wavelets in mathematical models has become increasingly popular in the recent years, and one can find root in the theory of distribution [23].

The rest of the paper is organized as follows. In Section 2, we give a brief retrospection to the quasi-wavelets based numerical method. In Section 3, we give a detailed description of spatial–temporal discretizations for Eq. (1), and detailed discrete formulations are given to the treatment of the clamped type condition (3). Section 4 is devoted to numerical experiments and discussion. This paper ends with a conclusion in Section 5.

2. The introduction of quasi-wavelets numerical method

Quasi-wavelets algorithm has emerged as a new promising local spectral collocation method for the solution of PDEs. Application of the quasi-wavelets algorithm to practical problem is a hot topic in applied mathematics and computational engineering. In this section, quasi-wavelets algorithm is briefly described. Shannon's delta sequence kernel is known as a wavelet scaling function [3], and it is one of the most important examples of the delta sequence kernel of Dirichlet type and is given by the following (inverse) Fourier transform of the characteristic function, $\chi_{[-\frac{\alpha}{2\pi},\frac{\alpha}{2\pi}]}$

$$\delta_\alpha(x) = \int_{-\infty}^{\infty} \chi_{[-\frac{\alpha}{2\pi},\frac{\alpha}{2\pi}]} e^{-i2\pi\eta x} d\eta = \frac{\sin(\alpha x)}{\pi x}, \tag{4}$$

where $\lim_{\alpha\to\alpha_0} \delta_\alpha(x) = \delta(x)$, α_0 is a generalized limit, $\delta(x)$ is a generalized function following from the fact that it is integrable inside a particular interval but has not a value in the interval. Numerically, Shannon's delta sequence kernel is also one of the most important cases, because for a given $\alpha > 0$, Shannon's delta sequence kernel provides a basis for the Paley–Wiener reproducing kernel Hilbert space $\mathbf{B}^2_\alpha$ [28], which is a subspace of the Hilbert space $\mathbf{L}^2(\mathbb{R})$. As such, an $\mathbf{L}^2(\mathbb{R})$ function $f(x)$ band-limited to α can be exactly reproduced as follows

$$f(x) = \int_{-\infty}^{+\infty} \delta_\alpha(x-y) f(y) dy = \int_{-\infty}^{+\infty} \frac{\sin(\alpha(x-y))}{\pi(x-y)} f(y) dy, \quad \forall f \in \mathbf{B}^2_\alpha, \tag{5}$$

here $\{x_k\}$ is an appropriate set of discrete points. However, in practical computations, the truncation error of Shannon's sampling formulae is substantial [14]. In this paper, we introduce the regularized Shannon's delta kernel [25]

$$\delta_{\Delta,\sigma}(x) = \frac{\sin(\pi x/\Delta)}{\pi x/\Delta} e^{-\frac{x^2}{2\sigma^2}}, \tag{6}$$