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A Finite Variable Difference Relaxation Scheme for hyperbolic–parabolic equations

Mayank Bajpayi, S.V. Raghurama Rao *

CFD Centre, Department of Aerospace Engineering, Indian Institute of Science, Bangalore, Karnataka 560012, India

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ABSTRACT

Using the framework of a new *relaxation system*, which converts a nonlinear viscous conservation law into a system of linear convection–diffusion equations with nonlinear source terms, a finite variable difference method is developed for nonlinear hyperbolic–parabolic equations. The basic idea is to formulate a finite volume method with an optimum spatial difference, using the Locally Exact Numerical Scheme (LENS), leading to a Finite Variable Difference Method as introduced by Sakai [Katsuhiro Sakai, A new finite variable difference method with application to locally exact numerical scheme, Journal of Computational Physics, 124 (1996) pp. 301–308.], for the linear convection–diffusion equations obtained by using a relaxation system. Source terms are treated with the well-balanced scheme of Jin [Shi Jin, A steady-state capturing method for hyperbolic systems with geometrical source terms, Mathematical Modeling Numerical Analysis, 35 (4) (2001) pp. 631–645]. Bench-mark test problems for scalar and vector conservation laws in one and two dimensions are solved using this new algorithm and the results demonstrate the efficiency of the scheme in capturing the flow features accurately.

1. Introduction

Numerical solution of hyperbolic and parabolic partial differential equations has reached a state of maturity in the past few decades. A major contribution to this progress came from the researchers in Computational Fluid Dynamics (CFD). The numerical solution of Euler equations of gas dynamics and the shallow water equations, which represent hyperbolic vector conservation laws, is now considered to be an established field, with several innovative numerical methods having been introduced in the past few decades. Some reviews of this history are available in [16,17,25,47,27]. One major focus in this development has been the introduction of higher order accurate methods which are stable and are free of numerical oscillations. The *Total Variation Diminishing* (TVD) schemes with limiters are especially designed for this purpose. However, difficulties still remain with this approach: getting uniformly higher order accuracy in all parts of the computational domain without clipping of the extrema, especially in multi-dimensions and with unstructured meshes, is hard to achieve and the research is still continuing in this area, as can be seen from the large number of papers continuously being published. The reader is referred to the books edited by Hussaini, van Leer and Rosendale [19], Barth and Deconink [5] and the references therein for a glimpse of these developments.

One of the essential difficulties associated with higher order schemes for convection dominated flow simulations is the nonmonotonicity of the solutions, manifesting as oscillations or wiggles, especially near high gradient regions or discontinuities. In this context, an important early development was the Godunov theorem [10] in which it was shown that linear

* Corresponding author. Tel.: +91 80 2293 3031; fax: +91 80 23600134.
E-mail addresses: raghu@aero.iisc.ernet.in, svraghuramarao@yahoo.com (S.V. Raghurama Rao).

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higher order schemes would necessarily lead to nonmonotone solutions. A way to circumvent this limitation is to make the coefficients nonlinear, by making the coefficients depend on the solution variables. The Flux Corrected Transport (FCT) algorithm of Boris and Book [6–8,53] and the Monotone Upstream Schemes for Conservation Laws (MUSCL) methodology of van Leer [48–51] are based on this strategy (see also [13]). Harten introduced the popular Total Variation Diminishing (TVD) schemes [14] by providing a rigorous mathematical foundation for this approach which was further developed by Sweby [46] and a lot of other researchers (see [19] for details). Goodman and Leveque [11] showed that multi-dimensional TVD schemes are no better than first order accurate schemes. Later, Essentially NonOscillatory (ENO) schemes were developed to overcome some of the deficiencies of the TVD schemes like clipping the extrema [15]. A detailed account of the further development of ENO schemes is given by Shu [44]. In spite of their advantages over TVD methods, ENO methods are not so convenient to extend to multi-dimensions. The development of higher order schemes is far from complete and the research in this area is still continuing; according to Roe [39], rigor is not yet to be found in all aspects of higher order schemes. In this context, it is worth looking for alternative approaches.

In the process of discretization of the convection dominated equations in CFD, it is considered advantageous to mimic the properties of the exact solutions of the original equations (when they are available) so that the resulting discretized equations have better chance to converge to the physically relevant solutions. For example, the upwind methods mimic the exact solutions of convection equations as closely as possible in a finite difference framework. An interesting alternative to this strategy is to develop a numerical method in which the coefficients in the difference equation satisfy the exact solution of the original convection (or convection–diffusion) equation. Some numerical methods using this strategy were developed by Allen and Southwell [1], Günther [12] and Sakai [40]. A related idea is used in *Nonstandard Finite Difference Methods* of Mickens [31] in which the difference equations have the same general solutions as the associated differential equations. An attempt to apply this approach to nonlinear convection equations and linear systems of convection equations is given in [52]. These methods produce results which are very close to the exact solutions, many times producing much superior results compared to traditional finite difference methods.

The Finite Variable Difference Method (FVDM) of Sakai [41] is one such interesting alternative to the traditional approaches. In this method, a nonoscillatory algorithm is developed for convection–diffusion equations, based on a variable mesh increment, with the optimal spatial difference being determined by minimizing the variance of the solution by choosing the roots of the difference equation to be nonnegative. Sakai [41,43] has demonstrated the efficiency of the FVDM for linear convection–diffusion problems and derived some new schemes based on this strategy for the 1-D Burgers equation, in which the hyperbolic terms are nonlinear. An important feature of the FVDM is that a nonoscillatory scheme is formulated explicitly based on the exact solution of convection–diffusion equations, a feature not shared by the convectional higher order schemes such as TVD methods. Note that in this scheme, the formulation of a nonoscillatory scheme is done directly for convection–diffusion equations, whereas most of the TVD methods are formulated only for convection equations. Another important feature of the FVDM is that the drive towards accuracy in developing a nonoscillatory scheme is based on minimizing the variance of the solution, variance being the deviation from the exact solution, which is more reasonable to use compared to the conventional derivation based on Taylor series expansions in the finite difference methods [41]. Yet another interesting feature of the FVDM is that formulation of the scheme is not based explicitly on artificial viscosity. Sakai [41] has demonstrated that the oscillations in the solutions are completely avoided by the FVDM in the case of steady equations and only very mild oscillations appear in the unsteady case. Because of all the above features, the FVDM is selected in this work, as an interesting alternative for study in developing accurate numerical methods for hyperbolic–parabolic partial differential equations representing conservation/balance laws.

Extending the FVDM directly to the unsteady and nonlinear Burgers equation seems to be nontrivial. So far, the FVDM has also not been applied to the hyperbolic systems of conservation laws. In this study, we extend Sakai's Finite Variable Difference Method to nonlinear Burgers equation, and also to a hyperbolic system of conservation laws (shallow water equations), by coupling this method to a new *relaxation system* which modifies the relaxation system of Jin and Xin [20], while linearizing the nonlinear hyperbolic–parabolic equations. We utilize the strategy used by Sakai, by applying FVDM to the Locally Exact Numerical Scheme in which the difference coefficients are determined such that the resulting difference equation satisfies the exact solution of the convection–diffusion equation, in our frame work of a novel relaxation system applied to the nonlinear convection–diffusion equations. This *Finite Variable Difference Relaxation Scheme* (FVDRS) is tested on some benchmark test problems for 1-d inviscid Burgers equation, 1-D viscous Burgers equation, 2-D inviscid Burgers equation, 2-D viscous Burgers equation and shallow water equations in both one and two dimensions. The results demonstrate the efficiency of this *Finite Variable Difference Relaxation Method*. It is worth noting that our method is not based on Riemann solvers, which are reported to be associated with a list of failures [34].

2. A relaxation system for viscous Burgers equation

Jin and Xin [20] introduced a relaxation system for hyperbolic equations like inviscid Burgers equation or Euler equations. A relaxation system provides a vanishing viscosity model for nonlinear hyperbolic conservation laws by replacing the nonlinear hyperbolic (convection) terms with linear hyperbolic terms with a stiff nonlinear source term which represents a mathematical relaxation process. The relaxation schemes, based on a relaxation system, are interesting alternatives to traditional schemes for solving hyperbolic conservation laws. The reader is referred to [20,32,2,26,36,37,4] for some numerical