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An operator splitting method for the Degasperis–Procesi equation

B.-F. Feng [a,*], Y. Liu [b,*]

[a] Department of Mathematics, The University of Texas-Pan American, Edinburg 78541, United States
[b] Department of Mathematics, The University of Texas-Arlington, United States

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ABSTRACT

An operator splitting method is proposed for the Degasperis–Procesi (DP) equation, by which the DP equation is decomposed into the Burgers equation and the Benjamin–Bona–Mahony (BBM) equation. Then, a second-order TVD scheme is applied for the Burgers equation, and a linearized implicit finite difference method is used for the BBM equation. Furthermore, the Strang splitting approach is used to construct the solution in one time step. The numerical solutions of the DP equation agree with exact solutions, e.g. the multipeakon solutions very well. The proposed method also captures the formation and propagation of shockpeakon solutions, and reveals wave breaking phenomena with good accuracy.

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1. Introduction

In this paper, we present an operator splitting method for the numerical solutions of the Degasperis–Procesi equation [19]

$$u_t + 3\kappa^3 u_x - u_{xxt} + 4uu_x = 3u_x u_{xx} + uu_{xxx}. \tag{1}$$

Degasperis and Procesi [19] studied a family of third order dispersive nonlinear equations

$$u_t - \alpha^2 u_{xxt} + \gamma u_{xxx} + c_0 u_x = (c_1 u^2 + c_2 u_x^2 + c_3 uu_{xx})_x, \tag{2}$$

with six real constants $c_0, c_1, c_2, c_3, \gamma, \alpha \in \mathbb{R}$. They found that there are only three equations were asymptotically integrable, i.e. the Korteweg-de Vries (KdV) equation ($\alpha = c_2 = c_3 = 0$), the Camassa–Holm (CH) equation ($c_1 = -\frac{3c_3}{2\alpha^2}, c_2 = \frac{c_3}{2}$), and one new equation ($c_1 = -\frac{2c_3}{\alpha^2}, c_2 = c_3$), which is named the Degasperis–Procesi (DP) equation later on.

The Camassa–Holm equation

$$u_t + 2\kappa^2 u_x - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \tag{3}$$

was first derived by Fokas and Fuchssteiner [24] as a bi-Hamiltonian system and then has attracted considerable attention since it was derived as a model equation for shallow water waves in 1993 [4]. The Camassa–Holm equation has been shown to be completely integrable [5]. Explicit form of multipeakon solutions for the Camassa–Holm equation was found by Beals et al. when $k \neq 0$ [2]. An approach based on the inverse scattering transform method (IST) provides an explicit form of the inverse mapping in terms of Wronskian [15].

The Degasperis–Procesi equation only differs from the Camassa–Holm equation by coefficients. Degasperis et al. proved the integrability of the DP equation by constructing a Lax pair and a bi-Hamiltonian structure [18]. These two equations share some common properties. They both can be viewed as the models of shallow water waves [4,5,31,16]. When $\kappa \neq 0$, the Camassa–Holm equation is related to the AKNS shallow water wave equation by a hodograph transformation [41],

* Corresponding author.
E-mail address: feng@utpa.edu (B.-F. Feng).

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and the Degasperis–Procesi equation is related to the Hirota-Satsuma shallow water wave equation by a similar hodograph transformation [37]. By use of the above findings, Matsuno obtained the multisoliton solutions of the DP equation when $\kappa \neq 0$ [37,38]. When $\kappa = 0$, both the CH and the DP equations have multipeakon solutions, the explicit form of multipeakon solution of the DP equation was found by Lundmark and Szmigielski by solving an inverse scattering problem of a discrete cubic string [34,35]. Furthermore, the peakon solutions for both the CH equation and the DP equation are orbitally stable [17,32].

On the other hand, although the DP equation has an apparent similarity to the CH equation, there are major structural differences between these two equations such as the Lax pair, wave breaking phenomena and the solutions. The isospectral problem in the Lax pair for the DP equation is the third-order equation [18], while the isospectral problem for the CH equation is the second order equation [4]. It is worth noting that Lundmark [36] showed that, when $\kappa = 0$, the DP equation has not only one peakon solution, $u(x,t) = ce^{-|x-ct|}$ but also a shock peakon solution of the form

$$u(x,t) = ce^{-|x-ct|} + \mathrm{sgn}(x-ct)\frac{s}{1+ts}e^{-|x-ct|}, \tag{4}$$

where $c, s (s > 0)$ are constants. Moreover, it is recently shown by Escher et al. [21] that the DP equation possesses a periodic shock wave solution given by

$$u(x,t) = \begin{cases} \left(\frac{\cosh\left(\frac{1}{2}\right)}{\sinh\left(\frac{1}{2}\right)}t + c\right)^{-1} \frac{\sinh\left(x-[x]-\frac{1}{2}\right)}{\sinh\left(\frac{1}{2}\right)}, & x \in \mathbb{R}\setminus\mathbb{Z},\ c > 0, \\ 0, & x \in \mathbb{Z}. \end{cases}$$

Lundmark further extended the multipeakon solution of the DP equation to multi-shockpeakon solution [36]

$$u(x,t) = \sum_{i=1}^{n} m_i(t)e^{-|x-x_i(t)|} + \sum_{i=1}^{n} s_i(t)\mathrm{sgn}(x-x_i)e^{-|x-x_i(t)|}, \tag{5}$$

where $m_i(t), x_i(t)$ and $s_i(t)$ stand for the momentum, position and strength of the ith shockpeakon. It is shown that (5) is a weak solution of the DP equation if and only if $m_i(t), x_i(t)$ and $s_i(t), i = 1, \ldots, n$ satisfy a system of ODEs ((2.4) and (2.5) in [36]). However, the integrability and the explicit form of above solution are still unclear even for $n = 2$ case. The only explicit form available is one shockpeakon solution mentioned above in (4).

Note that these peakons and shockpeakons are not the strong solutions in the Sobolev space $H^s, s \geqslant \frac{3}{2}$, but the global weak solutions in H^1[20]. Existence of these discontinuous (shock waves, [36]) solutions of the DP equation shows that the DP equation would behave radically different from the Camassa–Holm equation, but similar to the inviscid Burgers equation, which implies that a well-posedness theory should depend on some functional spaces which contain discontinuous functions. Indeed, this observation was confirmed by Coclite and Karlsen [11–13]. In [11–13], they proved the global existence and uniqueness of $L^1 \cap BV$ entropy weak solutions satisfying an infinite family of Kružkov-type entropy inequalities. and also proved existence of bounded weak solutions by an Oleĭnik-type estimate for L^∞ solutions to the DP equation with $\kappa = 0$.

For the purpose of numerical tests, the explicit form of two-peakon solution $u(x,t) = \sum_{i=1}^{2} m_i(t)e^{-|x-x_i(t)|}$ is listed here.

$$x_1(t) = \log\frac{(\lambda_1-\lambda_2)^2 b_1 b_2}{(\lambda_1+\lambda_2)(\lambda_1 b_1 + \lambda_2 b_2)}, \quad x_2(t) = \log(b_1+b_2), \tag{6}$$

$$m_1(t) = \frac{(\lambda_1 b_1 + \lambda_2 b_2)^2}{\lambda_1\lambda_2\left(\lambda_1 b_1^2 + \lambda_2 b_2^2 + \frac{4\lambda_1\lambda_2}{\lambda_1+\lambda_2}b_1 b_2\right)}, \quad m_2(t) = \frac{(b_1+b_2)^2}{\lambda_1 b_1^2 + \lambda_2 b_2^2 + \frac{4\lambda_1\lambda_2}{\lambda_1+\lambda_2}b_1 b_2}, \tag{7}$$

with $b_k(t) = b_k(0)e^{t/\lambda_k}$. Here λ_1, λ_2 are nonzero distinct constants, and $b_1(0)$ and $b_2(0)$ are two positive constants.

In the last decade, a lot of numerical schemes have been proposed for the Camassa–Holm equation. These include pseudo-spectral method [29], finite difference schemes [27,10], a finite volume method [1], a finite element method [43], multi-symplectic methods [14], a particle method based on the multipeakon solutions of the Camassa–Holm equation [6–8,28], an energy-conserving Galerkin scheme [39], and a self-adaptive mesh method based on an integrable semi-discretization of the Camassa–Holm equation [40,23]. On the contrary, the numerical methods available for the Degasperis–Procesi equation are only a few. Coclite et al. proposed several operator splitting schemes for the DP equation and proved convergence of those finite difference schemes to entropy weak solutions [10]. On the other hand, Hoel investigated entropy weak solutions of the DP equation numerically by a particle method based on the multi-shockpeakon solutions [26]. It is necessary to construct more effective numerical methods for the Degasperis–Procesi equation. The purpose of the present paper is to provide an operator splitting method for the numerical simulations of discontinuous solutions of the DP equation.

The remainder of the present paper is organized as follows. in Section 2, we present the operator splitting strategy, by which the Degasperis–Procesi equation is decomposed into the Burgers equation and the Benjamin–Bona–Mahony (BBM) equation. Then, extensive numerical experiments are performed in Section 3. These include peakon propagation and interactions, peakon–antipeakon interactions, shockpeakon–shockpeakon interactions, as well as initial value problems for some nonexact initial conditions. A good agreement is obtained in comparing exact and numerical solutions. In addition, the theoretical results of wave breaking phenomena are verified and explored numerically. Concluding remarks and comments are given in Section 4.