Available online at www.sciencedirect.com
ScienceDirect
Journal of Computational Physics 227 (2008) 3244–3259
JOURNAL OF COMPUTATIONAL PHYSICS
www.elsevier.com/locate/jcp

Isotropic compact interpolation schemes for particle methods ☆

Philippe Chatelain *,[1], Anthony Leonard

Graduate Aeronautical Laboratories, California Institute of Technology, Pasadena, CA 91125, USA

Received 11 April 2007; received in revised form 3 August 2007; accepted 26 November 2007
Available online 8 December 2007

Abstract

In particle methods, an accuracy degradation can occur because of the distortion of the element positions. A solution consists in the periodic re-initialization of the particles onto regular locations, at the nodes of a lattice. This so-called redistribution works by the interpolation of particle quantities. The present work considers the design of redistribution schemes on general lattices and in particular on lattices with a higher level of symmetry than the usual cubic lattice. Such lattices allow schemes which are more compact and more isotropic. We test our schemes in the context of three-dimensional vortex methods.

PACS: 02.60.Ed; 02.70.Ns; 47.11.+j; 47.32.Cc

Keywords: Interpolation; Particle methods; Vortex methods; Face-centered cubic lattice; Smoothed particle hydrodynamics

1. Introduction

In particle methods such as the vortex element method (VEM) or smoothed particle hydrodynamics (SPH), one is confronted with problems of integration and interpolation over the elements. In particular, this translates into accuracy degradation when the interpolating elements get too far apart in any direction (see [1,2,9]).

One can follow several approaches to tackle this problem. One approach consists in progressively introducing new elements in the domain. While elegant, this approach requires a costly algorithm to find the new elements' positions and strengths (see [4]).

The other approach is to build a whole new set of elements from the old ones. This process must take place every few time steps in order to prevent the particle distribution from getting too distorted. This so-called redistribution consists in interpolating the new strengths at the nodes of a new non-distorted lattice.

☆ Work sponsored by the US Department of Energy under contract No. DE-AC03-98EE50506.
* Corresponding author.
E-mail address: pchatela@inf.ethz.ch (P. Chatelain).
[1] Present address: Computational Science and Engineering Laboratory, ETH Zurich, CAB H83.1, Universitätstrasse 6, CH-8092 Zurich, Switzerland.

doi:10.1016/j.jcp.2007.11.039

This work considers the properties of redistribution schemes for a generic lattice and the construction of schemes which display high accuracy, isotropy and in some cases, compactness. We open this paper with a brief summary of the vortex element method in Section 2 as it constitutes a prototypical particle method and will be the basis of our numerical tests. We then discuss the properties of redistribution schemes (Section 3) and propose two approaches for the construction of isotropic schemes in Section 4. We consider lattices with a high degree of symmetry and derive a generalization of Monaghan [12] for the construction of high order schemes.

Our results are then applied to the design of schemes for lattices with high degrees of symmetry, such as the hexagonal and the face-centered cubic lattices.

We close this paper with the numerical application of these schemes in the context of vortex element methods. It is shown that they carry several advantages in terms of compactness and isotropy.

2. The vortex element method

We consider three-dimensional incompressible flow and the Navier–Stokes in vorticity form

$$\frac{D\boldsymbol{\omega}}{Dt} = (\nabla \mathbf{u}) \cdot \boldsymbol{\omega} + \nu \nabla^2 \boldsymbol{\omega}, \tag{1}$$

$$\nabla \cdot \mathbf{u} = 0, \tag{2}$$

where $\mathbf{u}(\mathbf{x}, t)$ is the velocity field, ν is the kinematic viscosity, and $\boldsymbol{\omega} = \nabla \times \mathbf{u}$ is the vorticity.

The vortex element method discretizes the vorticity field with particles which have positions $\mathbf{x}_i(t)$ and strengths $\boldsymbol{\alpha}_i(t) = \int_{V_i} \boldsymbol{\omega} d\mathbf{x}$. V_i is the particle volume. The field is then recovered through

$$\tilde{\boldsymbol{\omega}}(\mathbf{x}, t) = \sum_{i=1}^{N} \zeta_\epsilon(\mathbf{x} - \mathbf{x}_i(t)) \boldsymbol{\alpha}_i(t), \tag{3}$$

where ζ_ϵ is a smooth, usually radially symmetric, interpolating kernel.

The kernel smoothing radius ϵ determines the finest resolved scales. As a result, if the flow distorts the particle set and the inter-particle spacing grows beyond ϵ, the interpolation of Eq. (3) loses all accuracy and breaks down as some power of ϵ/h, where h is the particle spacing [1,2]. This so-called loss of overlap will affect any smoothed particle method.

A Helmholtz decomposition is used to represent the velocity field

$$\mathbf{u} = \nabla \phi + \nabla \times \boldsymbol{\psi}; \tag{4}$$

and we use the gauge $\nabla \cdot \boldsymbol{\psi} = 0$. The function ϕ is the scalar potential and the corresponding velocity is irrotational but potentially dilatational. This contribution will be kept at zero for the remainder of this paper. The stream-function is $\boldsymbol{\psi}$ which is related to the vorticity by the Poisson equation

$$\nabla^2 \boldsymbol{\psi} = -\boldsymbol{\omega}. \tag{5}$$

This Poisson problem can be solved through several techniques. In the present work, we use a Green's function approach

$$\boldsymbol{\psi}(\mathbf{x}) = \sum G(\mathbf{x} - \mathbf{x}_i) \boldsymbol{\alpha}_i(t). \tag{6}$$

The curl of $\boldsymbol{\psi}$ and its gradient then give us respectively the velocity and velocity gradient fields which are needed for the evolution equations of the particles positions and strengths. This problem is a N^2-complex problem, which can be made tractable with a fast multipole method [5,13,17].

In three dimensions, the particle discretization of the vorticity field $\tilde{\boldsymbol{\omega}}$ is not necessarily divergence-free. This spurious divergence needs to be kept at a low level. We also note that $\boldsymbol{\omega} = \nabla \times \mathbf{u}$ is clearly solenoidal; this quantity was used in the design of relaxation methods [16,10] and allows the definition of a divergence error

$$E_{\text{div}} = \int |\tilde{\boldsymbol{\omega}} - \boldsymbol{\omega}|^2 \, d\mathbf{x}. \tag{7}$$

In addition to relaxation, it has been shown [2] that the choice of the evaluation of the stretching term in Eq. (1) can alleviate this problem [2].