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High-order local absorbing conditions for the wave equation: Extensions and improvements

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Abstract

The solution of the time-dependent wave equation in an unbounded domain is considered. An artificial boundary $\mathcal{B}$ is introduced which encloses a finite computational domain. On $\mathcal{B}$ an absorbing boundary condition (ABC) is imposed. A formulation of local high-order ABCs recently proposed by Hagstrom and Warburton and based on a modification of the Higdon ABCs, is further developed and extended in a number of ways. First, the ABC is analyzed in new ways and important information is extracted from this analysis. Second, The ABCs are extended to the case of a *dispersive medium*, for which the Klein–Gordon wave equation governs. Third, the case of a *stratified medium* is considered and the way to apply the ABCs to this case is explained. Fourth, the ABCs are extended to take into account *evanescent modes* in the exact solution. The analysis is applied throughout this paper to two-dimensional wave guides. Two numerical algorithms incorporating these ABCs are considered: a standard semi-discrete finite element formulation in space followed by time-stepping, and a high-order finite difference discretization in space and time. Numerical examples are provided to demonstrate the performance of the extended ABCs using these two methods.

Keywords: Waves; High-order; Artificial boundary; Absorbing boundary condition; Higdon; Auxiliary variables; Finite elements; Finite differences; Wave guide; Dispersive waves; Stratified medium; Evanescent waves

1. Introduction

With the improvement of computational methods for the solution of wave problems in unbounded media, as encountered in geophysics, weather prediction, underwater acoustics, aeroacoustics, etc., the need for and interest in accurate schemes for treating artificial boundaries has increased in recent years. Among these

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methods, two have emerged as especially powerful [1]. The first one is the perfectly matched layer (PML) method, devised by Bérenger [2] in 1994 and since then further developed and used by many authors (see also [3]). The second type of method is that of high-order absorbing boundary conditions (ABCs). The use of ABCs has been very popular since the early 1970s – see the survey in [4] – but the recent development referred to here is the ability to use ABCs of an *arbitrarily high order*. The first such ABC has been devised by Collino [5] in 1993, and a few other formulations followed; see a survey in [6].

ABCs are boundary conditions imposed on artificial boundaries of computational domains. Given a wave problem in an unbounded medium, the infinite domain is truncated via an artificial boundary $\mathcal{B}$, thus dividing it into a finite computational domain Ω and a residual infinite domain D. A special boundary condition, called an ABC, is imposed on $\mathcal{B}$ in order to complete the statement of the problem in Ω (i.e. make the solution in Ω unique) and to ensure that no (or little) spurious wave reflection occurs from $\mathcal{B}$. The problem is then solved numerically in Ω. The setup is illustrated in Fig. 1 for a two-dimensional semi-infinite wave-guide. In this setup, which will serve as a prototype for this paper, $\mathcal{B} = \Gamma_{\mathrm{E}}$ is a cross-section of the wave-guide which constitutes the east side of Ω.

In theory, some of the classical ABCs, such as the Engquist-Majda ABCs [7] or the Bayliss–Turkel ABCs [8] can be defined up to any desired order. However, the appearance of increasingly high order derivatives in these ABCs renders them impractical beyond a certain order, typically 2 or 3. For example, the P-order Higdon ABC [9,10] involves P-order derivatives in space and time, and is thus very inconvenient for implementation when P is large. In fact, discrete Higdon conditions were developed in the literature, with the exception of [11], up to third order only.

There are two ways to construct practical ABCs with arbitrarily *high order* accuracy. The first way is to use an ABC based on a *nonlocal* operator; examples include the early work of Fix and Marin [12] and the Dirichlet-to-Neumann ABC [13], both in the frequency domain. However, in the time-domain nonlocal ABCs nonlocal conditions also require time convolutions. Although these can be treated efficiently in many cases [14], the nonlocal methods are inflexible in terms of the computational domain (e.g. rectangular boundaries cannot be used) and the governing equations (e.g. such methods are unavailable for stratified media.) The second type of high-order ABCs are those which make use of special *auxiliary variables*. The latter eliminate the need for any high-order derivatives; thus these ABCs can be practically employed with an arbitrarily high order [6,15]. One example is the high-order ABC of Hagstrom and Hariharan [16] for circular and spherical boundaries. Givoli and Neta [17,18] reformulated the Higdon ABC (in Cartesian coordinates) as a high-order ABC. Later, Hagstrom and Warburton [19] proposed a modification to the Givoli–Neta formulation with enhanced stability. The Hagstrom–Warburton (H–W) formulation is the basis for the present paper.

In [20] we compare the H–W formulation with the Givoli–Neta formulation from various aspects, and apply the former to exterior wave problems in two dimensions using a finite element scheme. In doing this we have to use special corner conditions at the four corners of the artificial boundaries. In [21] we concentrate on the question of choosing the computational parameters $0 < a_j \leqslant 1$, for $j = 0, 1, \ldots P$, which appear in the P-order H–W ABC and which signify cosines of incidence angles. A point of emphasis in [21] is that a comparison of boundary conditions based solely on the magnitude of reflection coefficients for propagating modes is a poor predictor of actual performance, particularly as the order is increased. This fact has been noted before, for example by Taflove and Hagness [22, Chapter 6]. In this reference it is suggested that the cause is wave speed mismatch resulting from discretization error. However, the highly-resolved calculations per-

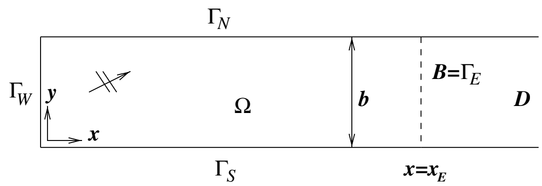

Fig. 1. A semi-infinite wave-guide, with an artificial boundary $\mathcal{B} = \Gamma_{\mathrm{E}}$ on which an ABC is to be applied.