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Fast sweeping method for the factored eikonal equation ☆

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ABSTRACT

We develop a fast sweeping method for the factored eikonal equation. By decomposing the solution of a general eikonal equation as the product of two factors: the first factor is the solution to a simple eikonal equation (such as distance) or a previously computed solution to an approximate eikonal equation. The second factor is a necessary modification/correction. Appropriate discretization and a fast sweeping strategy are designed for the equation of the correction part. The key idea is to enforce the causality of the original eikonal equation during the Gauss–Seidel iterations. Using extensive numerical examples we demonstrate that (1) the convergence behavior of the fast sweeping method for the factored eikonal equation is the same as for the original eikonal equation, i.e., the number of iterations for the Gauss–Seidel iterations is independent of the mesh size, (2) the numerical solution from the factored eikonal equation is more accurate than the numerical solution directly computed from the original eikonal equation, especially for point sources.

1. Introduction

The eikonal equation

$$|\nabla T|^2 = S^2(\mathbf{x}) \tag{1}$$

describes the traveltime $T(\mathbf{x})$ of a wave propagating with slowness (refraction index) $S(\mathbf{x})$ in space $\mathbf{x} \in \mathbb{R}^n$. In the case of anisotropic wave propagation, S depends additionally on $\nabla T/|\nabla T|$. When $S(\mathbf{x})$ is equal to one, the traveltime $T(\mathbf{x})$ corresponds to the distance function.

The eikonal equation plays an important role in many practical applications: computer vision, material science, computational geometry, etc. [16]. In seismic imaging, in particular, finite-difference solutions of the eikonal equation are used routinely for computing traveltime tables for numerical modeling and migration of seismic waves [22,20,17,10]. Although limited for computing only first-arrival traveltimes [7], eikonal solvers can be extended in several different ways to image multiple arrivals [3].

In this paper, we derive the factored eikonal equation by assuming that either an analytical or a numerical solution is available for Eq. (1) in the same domain but with different right-hand side. The solution $T(\mathbf{x})$ is then represented as a product of the known solution and an unknown factor, which satisfies the factored eikonal equation. The hope is that if $T(\mathbf{x})$ is some perturbation of the available solution, solving the factored eikonal equation either is easier or produces more accurate solution numerically.

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We develop a numerical algorithm based on the fast sweeping method to solve the factored eikonal equation and to evaluate the resultant gain in accuracy. The fast sweeping method (FSM) is an efficient iterative method that uses Gauss–Seidel iterations with alternating orderings to solve a wide range of Hamilton–Jacobi equations and other type of hyperbolic problems [4,27,19,26,8,9,25,15,14,24]. With an appropriate upwind scheme that captures the causality of the underlying partial differential equation, the iteration can converge in a finite number of iterations independent of the mesh size, which was proved for special cases in [26]. The intuition is the following: Information propagates along characteristics. Using a systematic alternating ordering strategy, all directions of characteristics can be divided into a finite number of groups and each group is covered simultaneously by one of the orderings. Moreover, any characteristics can be covered by a finite number of orderings [26]. With an appropriate upwind scheme that enforces the causality of the underlying partial differential equation, a Gauss–Seidel iteration propagates correct information in each updating along characteristics whose directions agree with the orderings.

After outlining the theory and the numerical algorithm, we conduct a series of numerical experiments, where numerical solutions are compared with analytical solutions for model problems. A significant improvement in accuracy is observed in comparison with FSM applied directly to the original eikonal equation. Finally, we apply our method to compute traveltime tables for the benchmark Marmousi model.

2. Factored eikonal equation

A fundamental property of Eq. (1) is that scaling slowness S by a constant corresponds to scaling traveltime T by the same constant. This property was used in seismic reflection imaging in the method of common-reflection-point scans [2,1].

Let us consider a factored decomposition

$$S(\mathbf{x}) = S_0(\mathbf{x})\,\alpha(\mathbf{x}), \tag{2}$$

$$T(\mathbf{x}) = T_0(\mathbf{x})\,\tau(\mathbf{x}) \tag{3}$$

and assume that

$$|\nabla T_0|^2 = S_0^2(\mathbf{x}). \tag{4}$$

If both T_0 and S_0 are known (either from an analytical solution or from a previous numerical computation), one can pose the problem of a numerical evaluation of the correction $\tau(\mathbf{x})$ on a computational grid with appropriate boundary conditions (see Remark 1 in the next section for assigning boundary conditions for τ). The function substitutions transform Eq. (1) to the *factored eikonal equation*

$$T_0^2(\mathbf{x})\,|\nabla\tau|^2 + 2\,T_0(\mathbf{x})\,\tau(\mathbf{x})\,\nabla T_0\cdot\nabla\tau + [\tau^2(\mathbf{x}) - \alpha^2(\mathbf{x})]\,S_0^2(\mathbf{x}) = 0. \tag{5}$$

When $\alpha(\mathbf{x})$ is a constant, the solution of Eq. (5) is trivial. When $\alpha(\mathbf{x})$ is not a constant but slowly varying, the hope is that accuracy of evaluating $T(\mathbf{x})$ from solving the factored eikonal equation can be greatly improved compared to a direct numerical solution of the original eikonal Eq. (1). One scenario is that point source singularities in the original solution $T(\mathbf{x})$ are well captured by $T_0(\mathbf{x})$. So the correction $\tau(\mathbf{x})$ is a smooth function in a neighborhood of the point sources. For example, when computing the traveltime for a point source the solution is singular at the source. Special treatment, such as using local grid refinement near the source, has to be implemented in order to achieve high order accuracy for the numerical solution to the eikonal equation [13]. However, locally the singularity of the solution to a regular eikonal equation (assuming $S(\mathbf{x})$ is smooth and strictly positive) at a point source is the same as the singularity of the distance function to that point source up to a smooth modification. Numerical tests in Section 4 show that the numerical solution based on the factored eikonal equation can be significantly more accurate than the numerical solution computed directly from the original eikonal equation. Also note that although ∇T_0 does not exist at the source point, it is well-defined away from the source point. It provides good approximation of all ray directions near the point source. This is crucial for computing accurate solutions away from the point source, which cannot be approximated easily on a discrete mesh.

When taking $S_0(\mathbf{x}) = 1, T_0(\mathbf{x})$ is the distance function. In the case of simple domains and boundary conditions, the distance can be evaluated analytically. For example, the distance from a point source at $\mathbf{x_0}$ is $T_0(\mathbf{x}) = |\mathbf{x} - \mathbf{x_0}|$, which transforms Eq. (5) to

$$|\mathbf{x} - \mathbf{x_0}|^2\,|\nabla\tau|^2 + 2\,\tau(\mathbf{x})\,(\mathbf{x} - \mathbf{x_0})\cdot\nabla\tau + \tau^2(\mathbf{x}) - S^2(\mathbf{x}) = 0. \tag{6}$$

A numerical solution of Eq. (6) was investigated previously in geophysical applications [12,23]. Simple analytical solutions exist for several other particular cases of slowness distributions such as a constant gradient of the slowness squared, a constant gradient of the velocity (inverse slowness), etc. [5].

Remark. In general T may have other singularities, e.g., due to the intersections of different characteristics in addition to point singularities at source points. Hence τ may also have singularities away from sources. In practice it is impossible to know the exact locations of these singularities without knowing the exact solution. However, for numerical computation (especially if the scheme is upwind), singularities caused by the intersections of different characteristics are less damaging or polluting than point singularities at source points from where characteristics emanate. In principle we only need to choose a