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Adaptive mesh refinement based on high order finite difference WENO scheme for multi-scale simulations

Chaopeng Shen [a,*,1], Jing-Mei Qiu [b], Andrew Christlieb [c]

[a] Department of Civil and Environmental Engineering, Michigan State University, East Lansing, 48824, United States
[b] Department of Mathematical and Computer Science, Colorado School of Mines, Golden, 80401, United States
[c] Department of Mathematics, Michigan State University, East Lansing, 48824, United States

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ABSTRACT

In this paper, we propose a finite difference AMR-WENO method for hyperbolic conservation laws. The proposed method combines the adaptive mesh refinement (AMR) framework [4,5] with the high order finite difference weighted essentially non-oscillatory (WENO) method in space and the total variation diminishing (TVD) Runge–Kutta (RK) method in time (WENO-RK) [18,10] by a high order coupling. Our goal is to realize mesh adaptivity in the AMR framework, while maintaining very high (higher than second) order accuracy of the WENO-RK method in the finite difference setting. The high order coupling of AMR and WENO-RK is accomplished by high order prolongation in both space (WENO interpolation) and time (Hermite interpolation) from coarse to fine grid solutions, and at ghost points. The resulting AMR-WENO method is accurate, robust and efficient, due to the mesh adaptivity and very high order spatial and temporal accuracy. We have experimented with both the third and the fifth order AMR-WENO schemes. We demonstrate the accuracy of the proposed scheme using smooth test problems, and their quality and efficiency using several 1D and 2D nonlinear hyperbolic problems with very challenging initial conditions. The AMR solutions are observed to perform as well as, and in some cases even better than, the corresponding uniform fine grid solutions. We conclude that there is significant improvement of the fifth order AMR-WENO over the third order one, not only in accuracy for smooth problems, but also in its ability in resolving complicated solution structures, due to the very low numerical diffusion of high order schemes. In our work, we found that it is difficult to design a robust AMR-WENO scheme that is both conservative and high order (higher than second order), due to the mass inconsistency of coarse and fine grid solutions at the initial stage in a finite difference scheme. Resolving these issues as well as conducting comprehensive evaluation of computational efficiency constitute our future work.

1. Introduction

Recently high order (higher than second order) methods have attracted increasing attention from many computational fields of study. High order schemes are often considered expensive, unstable and unnecessary. However, it is argued in some studies that the expenses of high order schemes are compensated by their high order accuracy and very low numerical

* Corresponding author.
E-mail addresses: cshen@lbl.gov (C. Shen), jingqiu@mines.edu (J.-M. Qiu), Christlieb@math.msu.edu (A. Christlieb).
[1] Now at: Computational Research Division, Lawrence Berkeley National Laboratory, Berkeley, CA 94720, United States. Tel.: +1 510 486 7286.

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diffusion, because high order methods can resolve solution structures with relatively coarse numerical meshes. As the error tolerance for the problem gets very small, or the integration time gets very long in a time-dependent problem, high order schemes often become more efficient than lower order ones [27]. Shu [29] demonstrated that high order schemes can resolve solution structures which are impractically expensive to obtain if a low order scheme is used, specifically when the solution structures become very complicated in the long time setting. However, the accusation that high order schemes are still more or less valid, as more computational dollars per node are needed by a higher order scheme. One can not use an arbitrarily coarse grid because the mesh needs to be fine enough to resolve the physical phenomena of interest. Popular classes of high order methods for hyperbolic equations include the finite difference or finite volume WENO method [31,18], the finite element discontinuous Galerkin (DG) method [11,12], the spectral method [15], the space time conservative element and the solution element method [8,9] and so on.

Adaptive mesh refinement (AMR) was originally developed by Berger et. al. [5,6]. The method refines the mesh locally to focus computational effort where it is most needed. AMR has found popularity in a wide range of fields such as computational fluid dynamics, astrophysics, oceanography, biophysics and many others [26]. In particular, it has been shown that the AMR method is advantageous for physical systems with vastly different spatial scales. For example, the evolution of a hyperbolic equation often leads to local shocks, near which numerical methods can have large errors. These errors might propagate and further contaminate the solution across the entire domain. Moreover, the solution structures of different scales might interact, and the failure to address features on one scale can jeopardize the quality of the entire solution. The multi-resolution nature of the AMR algorithm makes it ideal for these cases, where computational savings can be enormous [5]. There have been many efforts aimed at further developing the AMR algorithm, for example, for high dimensional problems [3], for use with unstructured meshes [23], for coupling with high order integrators [6,27,1,22], and for elliptic equations [2]. A detailed comparison of AMR versus high order schemes is given in [17] for a range of different problems. It was concluded in [17] that the AMR scheme will be advantageous if the AMR region is below a certain proportion of the entire domain; and that it is advantageous to have an AMR scheme with as high order as the regularity of the PDE.

Given the above mentioned features of the AMR algorithm, it is natural to combine the AMR algorithm with numerical schemes that are highly accurate, robust and efficient. There have been many research efforts in this direction. Sebastian & Shu [28] introduced a multi-domain WENO scheme but the refinement region was fixed and therefore not adaptive. Li [22] combined high order finite difference WENO with AMR. However, a linear prolongation in space was used, which reduces the scheme to second order accurate. Further, the scheme in [22] does not have temporal refinement, thus the temporal step size is limited by the CFL restriction of the finest mesh. Baeza [1] also described an AMR-WENO scheme, but the method was only second order accurate in time. There has also been much effort in designing adaptive versions of the DG and spectral methods for hyperbolic problems [14,16,34,24,33,27]. In the present work, we will focus on designing an adaptive finite difference WENO method. Compared with the DG and spectral methods, the finite difference WENO methods, when combined with the total variation diminishing (TVD) Runge–Kutta (RK) method in time, are more robust in resolving discontinuous shock structures.

In this paper, we couple the AMR framework with a high order finite difference WENO-RK scheme. The finite difference WENO-RK scheme is well-known for its high order accuracy, robustness and efficiency, as well as its straightforward extension to multi-dimensional problems (dimension by dimension) compared with finite volume schemes. There are a few major roadblocks that need to be cleared in order to realize a high order coupling. First, it is difficult to maintain high order accuracy across several levels of grids. In the AMR algorithm, the fine mesh solutions are interpolated from the coarse mesh solutions. Solutions at boundary points are needed not only at the current time but also at intermediate fine grid time steps (subcycles) as well as RK sub-stages. The same temporal and spatial accuracy are required in the data prolongation procedures as those of the base integrator. Secondly, introducing a high order accurate scheme in the AMR setting makes it harder to maintain local mass conservation. Thirdly, robust and inexpensive refinement criteria are difficult to find. In the original AMR [6], the refinement criterion relies on Richardson extrapolation (RE). In the finite difference WENO method, nonlinear WENO weights could be used as a measure of solution smoothness, hence providing an inexpensive refinement criterion. Alternatively, the difference between solutions on a coarse and fine grid can be used as a refinement criterion. Karni [19] also suggested a smoothness indicator based on weak local truncation error of the numerical solution, which can be used as a refinement criterion.

In the discussion that follows, we focus on the high order aspect of the proposed AMR-WENO algorithm. We compare different implementations of the AMR-WENO scheme, e.g. with integrators of different orders (third and fifth order), with different refinement criteria (RE and WENO weights), and with different data restriction procedures (a point value replacement and a conservative update). The classical AMR procedure is briefly reviewed in Section 2. We describe the proposed AMR-WENO scheme in Section 3. In order to describe our algorithm clearly, most of our discussions are based on a 1D hyperbolic equation,

$$u_t + f(u)_x = 0. \tag{1}$$

The 1D algorithm can be generalized to high-dimensional problems without much technical difficulty. In Section 4, computational results for several 1D and 2D hyperbolic problems are demonstrated, illustrating the high order accuracy, robustness and efficiency of the proposed AMR-WENO scheme. Conclusions are given in Section 5.