Journal of Computational Physics 230 (2011) 4011–4027

Contents lists available at ScienceDirect

Journal of Computational Physics

journal homepage: www.elsevier.com/locate/jcp

Improvements on open and traction boundary conditions for Navier–Stokes time-splitting methods

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ARTICLE INFO

Article history:
Received 5 November 2010
Received in revised form 4 February 2011
Accepted 16 February 2011
Available online 22 February 2011

Keywords:
Navier–Stokes equations
Projection methods
Pressure-correction methods
Fractional step methods
Traction boundary condition
Open boundary condition
Pressure boundary condition
Bifurcated tube
Square cylinder

ABSTRACT

We present in this paper a numerical scheme for incompressible Navier–Stokes equations with open and traction boundary conditions, in the framework of pressure-correction methods. A new way to enforce this type of boundary condition is proposed and provides higher pressure and velocity convergence rates in space and time than found in the present state of the art. We illustrate this result by computing some numerical and physical tests. In particular, we establish reference solutions of a laminar flow in a geometry where a bifurcation takes place and of the unsteady flow around a square cylinder.

1. Introduction

The main difficulty in obtaining the numerical solution of the incompressible Navier–Stokes equations, apart from the treatment of non-linearities, lies in the Stokes stage and specifically in the determination of the pressure field which will ensure a solenoidal velocity field. The question is how to uncouple the velocity and the pressure operators to efficiently reach an accurate solution to the unsteady Stokes problem, without degrading the predefined stability properties of the chosen scheme for the Navier–Stokes equations.

Historically, the first idea was proposed by Uzawa [1,2] and applied for numerical approximations with several methods [3,4]. It is a safe and efficient method for the numerical approximation of the Stokes problem. In complex geometries or three-dimensional domains, this method turns out to be inappropriate for its computational time cost which becomes very high. A different method is to uncouple the pressure from the velocity by means of a time splitting scheme that significantly reduces the computational cost. A large number of theoretical and numerical works have been published that discuss the accuracy and the stability properties of such methods. The most widespread methods are pressure-correction schemes. They were first introduced by Chorin-Temam [5,6], and improved by Goda (the standard incremental scheme) in [7], and later by Timmermans in [8] (the rotational incremental scheme). They require the solution of two sub-steps for each time step: the pressure is treated explicitly in the first one, and is corrected in the second one by projecting the predicted velocity onto an ad hoc space. The governing equation on the pressure or the pressure increment is a Poisson equation derived from the

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doi:10.1016/j.jcp.2011.02.024

momentum equation by requiring incompressibility. In [9,10], the authors proved the reliability of such approaches from the stability and the convergence rate point of view. A series of numerical issues related to the analysis and implementation of fractional step methods for incompressible flows are addressed in the review paper of Guermond et al. [11]. In this reference, the authors describe the state of the art for both theoretical and numerical results related to the time splitting approach. One emerging conclusion points out that time splitting can be a high order alternative to solve the unsteady Stokes problem when the velocity boundary condition is of the Dirichlet type.

However, in many applications such as free surface problems and channel flows, one also has to deal with an outlet boundary condition on all or part of the boundary, on which the applied numerical condition should not disturb upstream flow. A large variety of this kind of boundary condition exists [12,13], such as the non-reflecting outlet boundary condition (and its adaptations) derived from a wave equation, which is suited to wake and jet flow with moderate and high Reynolds number [14–19]. Hereafter we will present some improvements on the open or traction boundary condition which is efficient for low Reynolds number and fluid–structure interactions [20–22]. The traction boundary condition was successfully used to compute various flows such as those around a circular cylinder, over a backward facing step and in a bifurcated tube [20]. Bruneau [23] proposes an evolution of the traction boundary condition involving inertial terms. Hasan [24] proposes, in the computation of incompressible flow around rigid bodies, to extrapolate velocity on the outflow boundary, pressure being obtained through traction boundary conditions.

In the case of open or traction boundary conditions, several questions remain open especially when the pressure-correction version is considered as mentioned in [10]. Indeed, Guermond et al. [11], have proven that only spatial and time convergence rates between $O(\Delta x+\Delta t)$ and $O(\Delta x^{3/2}+\Delta t^{3/2})$ on the velocity and $O(\Delta x^{1/2}+\Delta t^{1/2})$ on the pressure are to be expected with the standard incremental scheme, and between $O(\Delta x+\Delta t)$ and $O(\Delta x^{3/2}+\Delta t^{3/2})$ on the velocity and pressure for the rotational incremental scheme. Février̀e [25] combines the penalty and projection methods to improve error levels of a manufactured case with open boundary conditions, but without improvement of the convergence rate. Finally, Liu [20], with a pressure Poisson equation formulation, proposes a new implementation of the open and traction boundary conditions. He proves unconditional stability with a first order time scheme and shows second order numerical convergence rate on velocity and pressure.

The aim of this paper is to propose a numerical scheme for the incompressible Navier–Stokes equations with open and traction boundary conditions, using the pressure-correction version of the time splitting methods. A new way to enforce this type of boundary condition is proposed and improves the order of convergence for both pressure and velocity. In the second part of this article we will describe the governing equations, and, in the third part, the pressure-correction schemes for open boundary conditions. In the fourth part, we will present the improvements we made on the numerical implementation of the traction and open boundary conditions. In a fifth section we will illustrate numerically the proposed method with two manufactured cases and two physical cases.

First of all we specify some notations. Let us consider a Lipschitz domain $\Omega\subset\mathbb{R}^d$, $(d=2$ or $3)$, the generic point of Ω is denoted $\mathbf{x}$. The classical Lebesgue space of square integrable functions $L^2(\Omega)$ is endowed with the inner product:

$$(\phi,\psi)=\int_\Omega \phi\psi\ d\boldsymbol{x},$$

and the norm

$$\|\psi\|_{L^2(\Omega)}=\left(\int_\Omega |\psi(\boldsymbol{x})|^2\right)^{\frac{1}{2}}.$$

We break the time interval $[0,t^*]$ into N subdivisions of length $\Delta t=\frac{t^*}{N}$, called the time step, and define $t^n=n\Delta t$, for any n, $0\leqslant n\leqslant N$. Let $\varphi^0,\varphi^1,\dots,\varphi^N$ be some sequence of functions in $E=L^2$. We denote this sequence by $\varphi^{\Delta t}$, and we define the following discrete norm

$$\|\varphi^{\Delta t}\|_{l^2(E)}=\left(\Delta t\sum_{k=0}^{N}\|\varphi^k\|_E^2\right)^{\frac{1}{2}}. \tag{1.1}$$

In practice the following error estimator can be used

$$\|\varphi\|^2_{E(t^*)}=\|\varphi(\cdot,t^*)\|_E. \tag{1.2}$$

Finally, bold Latin letters like $\boldsymbol{u}$, $\boldsymbol{w}$, $\boldsymbol{f}$, etc., indicate vector valued quantities, while capitals (e.g. $\boldsymbol{X}$, etc.) are functional sets involving vector fields.

2. Governing equations

Let Ω be a regular bounded domain in $\mathbb{R}^d$ with $\boldsymbol{n}$ the unit normal to the boundary $\Gamma=\partial\Omega$ oriented outward. We suppose that Γ is partitioned into two portions Γ_D and Γ_N.

Our study consists, for a given finite time interval $[0,t^*]$ in computing velocity $\boldsymbol{u}=\boldsymbol{u}(\boldsymbol{x},t)$ and pressure $p=p(\boldsymbol{x},t)$ fields satisfying: