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Error analysis of multipoint flux domain decomposition methods for evolutionary diffusion problems

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ABSTRACT

We study space and time discretizations for mixed formulations of parabolic problems. The spatial approximation is based on the multipoint flux mixed finite element method, which reduces to an efficient cell-centered pressure system on general grids, including triangles, quadrilaterals, tetrahedra, and hexahedra. The time integration is performed by using a domain decomposition time-splitting technique combined with multiterm fractional step diagonally implicit Runge–Kutta methods. The resulting scheme is unconditionally stable and computationally efficient, as it reduces the global system to a collection of uncoupled subdomain problems that can be solved in parallel without the need for Schwarz-type iteration. Convergence analysis for both the semidiscrete and fully discrete schemes is presented.

1. Introduction

Let us consider the parabolic initial-boundary value problem

$$p_t + \nabla\cdot\mathbf{u} = f \quad \text{in } \Omega\times(0,T], \tag{1a}$$

$$\mathbf{u} = -K\nabla p \quad \text{in } \Omega\times(0,T], \tag{1b}$$

$$p = g \quad \text{on } \Gamma_D\times(0,T], \tag{1c}$$

$$\mathbf{u}\cdot\mathbf{n} = 0 \quad \text{on } \Gamma_N\times(0,T], \tag{1d}$$

$$p = p_0 \quad \text{in } \Omega\times\{0\}, \tag{1e}$$

where $\Omega\subset\mathbb{R}^d$, $d=2$ or 3, is a convex polygonal or polyhedral domain with Lipschitz continuous boundary $\partial\Omega = \overline{\Gamma}_D\cup\overline{\Gamma}_N$ such that $\Gamma_D\cap\Gamma_N=\emptyset$. In this formulation, $p=p(\mathbf{x},t)$, $\mathbf{u}=\mathbf{u}(\mathbf{x},t)$, $f=f(\mathbf{x},t)$, $g=g(\mathbf{x},t)$ and $p_0=p_0(\mathbf{x})$. Further, $\mathbf{n}$ is the outward unit normal on $\partial\Omega$ and $K=K(\mathbf{x})$ is a symmetric and uniformly positive definite tensor satisfying, for some $0<\kappa_*\leqslant\kappa^*<\infty$,

$$\kappa_*\xi^T\xi \leqslant \xi^T K(\mathbf{x})\xi \leqslant \kappa^*\xi^T\xi \quad \forall\mathbf{x}\in\Omega,\ \forall\xi\in\mathbb{R}^d. \tag{2}$$

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In applications to flow in porous media, $\mathbf{u}$ is the Darcy velocity, p is the fluid pressure, and K represents the rock permeability divided by the fluid viscosity.

In this work, we extend the formulation of the multipoint flux mixed finite element (MFMFE) method (cf. [1–7]) to time-dependent diffusion problems. In doing so, we introduce an unconditionally stable domain decomposition time-splitting technique which is designed to take advantage of the computational efficiency of parallel implementations. Similar ideas have been proposed in [8] and [9], where alternative compatible spatial discretizations – such as mimetic finite difference or expanded mixed finite element methods – have been studied in combination with fractional step time integrators. The convergence analysis developed in these works is restricted to the use of rectangular grids and diagonal tensors K. Here, we propose a general formulation which overcomes these restrictions and applies to two- and three-dimensional meshes composed of simplices, quadrilaterals and hexahedra, with full tensor coefficients.

The MFMFE scheme was motivated by the multipoint flux approximation (MPFA) methods (cf. [10–13]), where the introduction of sub-face (sub-edge in 2D) fluxes allows for local flux elimination and reduction to a cell-centered pressure scheme. In the MFMFE framework, similar elimination is achieved by employing appropriate finite element spaces and special quadrature rules. The MFMFE method is based on the lowest order Brezzi–Douglas–Marini, $\mathcal{BDM}_1$, or Brezzi–Douglas–Durán–Fortin, $\mathcal{BDDF}_1$, spaces (cf. [14] and [15], respectively), with a trapezoidal quadrature rule applied on the reference element (cf. [1,5–7]; see also [2–4] for an alternative formulation on quadrilateral grids using a broken Raviart–Thomas space).

The system of ordinary differential equations resulting from the spatial semidiscretization process is integrated in time by means of an operator splitting method (see [16]). To this end, the time derivative function is first partitioned via an overlapping domain decomposition splitting technique. This kind of splitting was first introduced in [17,18] in the context of regionally-additive schemes, and has been subsequently extended in [19–21] for solving linear parabolic problems. The monographs [22,23] show an overview of some recent contributions to the topic. As a matter of fact, the domain decomposition operator splitting requires a time integrator which allows for multiterm partitioning. A suitable choice for such a method can be found within the class of m-part fractional step Runge–Kutta (FSRK$_m$) schemes. These time integrators are constructed by merging together m diagonally implicit Runge–Kutta schemes into a single composite method. A survey on their use for solving linear parabolic problems can be found in [24]. Here, we present the general formulation of this class of schemes, and subsequently focus on a particular family of them, first proposed in [25], which extends the classical Peaceman–Rachford alternating direction implicit (ADI) method (cf. [26]) to a domain decomposition partitioning into an arbitrary number of terms. Similar combinations of the Peaceman–Rachford procedure with domain decomposition methods have been studied in [27]; however, the method there is based on non-overlapping decompositions in the context of elliptic problems and is restricted to an operator splitting into two split terms. For a related work on non-overlapping domain decompositions, see [28]. The key to the efficiency of our proposed method lies in reducing the system matrix to a collection of uncoupled submatrices of lower dimension. As compared to classical domain decomposition algorithms (cf. [29]), this technique does not involve any Schwarz iteration procedure, thus reducing the computational cost of the overall solution process.

We note that, to suitably merge the space and time discretization methods, the standard definition of the split functions for the scalar variable is no longer valid. In this case, we need to introduce a specific splitting for the flux variable that properly handles the degrees of freedom defined in the MFMFE approach.

We derive a priori error estimates for both the continuous-in-time and fully discrete formulations of the problem under study. Two variants of the MFMFE method are analyzed, namely: a symmetric scheme (cf. [1,2,4–6]), which applies to simplices and $\mathcal{O}(h^2)$-perturbations of parallelograms and parallelepipeds; and a non-symmetric method (cf. [3,7]), designed to preserve the accuracy on general quadrilaterals and hexahedra. The semidiscrete scheme is proved to satisfy the following convergence properties. In the symmetric case, the velocity and pressure variables are $\mathcal{O}(h)$ convergent, the latter being $\mathcal{O}(h^2)$ superconvergent at the cell centers. In the non-symmetric case, the velocity variable is shown to be $\mathcal{O}(h)$ convergent either when compared to a suitable projection of the true solution or when computed in a face-based (edge-based) norm; in turn, the pressure preserves its $\mathcal{O}(h)$ optimal convergence. After a suitable elimination procedure for the velocities, we obtain a fully discrete scheme in the pressure variable. The convergence analysis is described in detail for the symmetric method and just sketched for its non-symmetric counterpart. Based on stability and consistency results, unconditional convergence of order $\mathcal{O}(h+\tau^2)$ and superconvergence of order $\mathcal{O}(h^2+\tau^2)$ are obtained, respectively, in the continuous and discrete L^2-norm in space. These results extend those derived in [30] for the Peaceman–Rachford ADI method applied to linear problems to a domain decomposition splitting formula into an arbitrary number of split terms. From a computational viewpoint, parallel scalability tests for the non-iterative time-splitting technique are reported for the first time.

The rest of the paper is organized as follows. In Section 2, we describe the MFMFE semidiscrete scheme. Some convergence results for the elliptic problem are presented in Section 3. Based on these results, the error analysis is extended to the parabolic problem in Section 4. Section 5 introduces a domain decomposition splitting method leading to the fully discrete scheme. The convergence analysis for such a scheme is provided in Section 6. Finally, a series of numerical experiments illustrates the convergence and scalability behavior of the proposed algorithms in Section 7. A specific application to transient flow modeling in heterogeneous porous media is also discussed.