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Physics-compatible discretization techniques on single and dual grids, with application to the Poisson equation of volume forms

Artur Palha [a], Pedro Pinto Rebelo [b], René Hiemstra [c], Jasper Kreeft [d], Marc Gerritsma [b,*]

[a] Delft University of Technology, Faculty of Aerospace Engineering, Wind Energy Group, P.O. Box 5058, 2600 GB Delft, The Netherlands
[b] Delft University of Technology, Faculty of Aerospace Engineering, Aerodynamics Group, P.O. Box 5058, 2600 GB Delft, The Netherlands
[c] ICES, Institute for Computational Engineering and Sciences, 201 East 24th St, Austin, TX, USA
[d] Shell Global Solutions, The Netherlands

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ABSTRACT

This paper introduces the basic concepts for physics-compatible discretization techniques. The paper gives a clear distinction between vectors and forms. Based on the difference between forms and pseudo-forms and the $\star$-operator which switches between the two, a dual grid description and a single grid description are presented. The dual grid method resembles a staggered finite volume method, whereas the single grid approach shows a strong resemblance with a finite element method. Both approaches are compared for the Poisson equation for volume forms. By defining a suitably weighted inner product for 1-forms this approach can readily be applied to anisotropic diffusion models for volume forms.

1. Introduction

Mimetic methods aim to preserve essential physical/mathematical structures in a discrete setting. Many of such structures are *topological*, i.e. independent of metric, and involve *integral relations*. Since integration will play an important role and integration of differential forms is a metric-free operation, we will work with differential forms. Formally, differential forms are linear functionals on multi-vectors, but Flanders, [18, p. 1], refers to them as '*things which occur under integral signs*'. Such would not be the case if we were to use vectors, because integration of vector quantities is a metric operation. The same holds for vector operations: The grad, curl and div are metric-dependent operators, whereas the exterior derivative, which plays a similar role for differential forms, is metric-free. The important difference between vectors and forms will be explicitly addressed in this paper.

When integrals over k-dimensional geometric objects are considered, the orientation of these k-dimensional objects needs to be taken into account. If we change the orientation of a point, curve, surface or volume, some integral values change sign, whereas others do not. For instance the work W_{AB} of a conservative force along a curve γ connecting the points A and B is equal to $-W_{BA}$, i.e. the work of the same force in the opposite direction along the curve. So the physical

* Corresponding author.
E-mail addresses: A.Palha@TUDelft.nl (A. Palha), P.J.PintoRebelo@TUDelft.nl (P.P. Rebelo), rrhiemstra@utexas.edu (R. Hiemstra), Jasper.Kreeft@Shell.com (J. Kreeft), M.I.Gerritsma@TUDelft.nl (M. Gerritsma).

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quantity work changes sign when we change the orientation of the curve. Mass, on the other hand, which is the integral of mass density over a volume, does not change sign when we change the orientation.

We need to consider two distinct types of differential forms: Those that do not change sign when the orientation of volumes is reversed, the *true forms* and those that do change sign, the *pseudo-forms*. The operator which switches between forms and pseudo-forms is called the *Hodge-⋆ operator*. This operator depends explicitly on the metric.

Integrals and integral relations can be represented without error in terms of duality pairing between chains and cochains. The distinction between integrals of true forms and pseudo-forms requires in principle two grids: One on which we represent the integral of a true form and the other grid on which we represent the integral of a pseudo-form. The formulation obtained by employing two dual grids resembles staggered finite volume methods.

An alternative way to implement the action of the Hodge-⋆ operator is to make use of an inner product. In this approach only one grid is required. The formulation based on a single grid approach leads to a finite element method.

In this paper we try to explain this structure in more detail and show in a specific example that the single grid approach and the dual grid approach lead to almost identical solutions. By making the clear distinction between topological concepts (integrals and discrete integral representations) and metric-dependent operations (Hodge-⋆), it is very easy to switch between orthogonal grids and curvilinear grids.

Commuting relations between the discretization (mimetic projection) and operations at the continuous and discrete level will play an important role in order to ensure that the 'discrete system behaves just like the continuous system'.

Throughout this paper the basic idea will be highlighted by putting statements in a box and the main idea of this paper is:

> A discrete representation of a physical system will display the same structure/dynamics when discrete operators and continuous operators commute with the projection of the infinite dimensional space onto the discrete space.

The same idea has been put forward in many different papers. Early work in this field was reported by Branin, [9]. Dissecting physical models into metric-free components and metric-dependent parts was originally proposed by Tonti, [50]. Application of Tonti's ideas to electromagnetism is fully treated by Mattiussi [34]. A very good introduction in the geometric structure of electromagnetism is given by Bossavit, [6,7]. But based on the analogies described by Tonti, the construction advocated by Bossavit has a much wider range of applicability than just electromagnetism. Hodge theory of harmonic forms was described by Dodziuk, [17]. Hyman and Scovel, [24], derived mimetic operators in a finite difference setting which were later generalized by Bochev and Hyman, [4]. Mimetic finite difference methods are described in Brezzi et al. and Steinberg et al., [11,25,26,44,46], the extensive paper by Lipnikov et al., [33] and Bonelle and Ern, [5]. Arnold, Falk and Winther have described an extensive framework in a finite element context, [1,2]. The geometric ideas underlying this paper are also extensively studied by Desbrun et al., [15] and Hirani, [23] and DiCarlo et al., [16]. For fluid flow calculations, mimetic methods were used by Perot, [40,41] and the importance of preserving physical invariants was illustrated in Perot's review paper [39]. Application of these ideas to spectral elements was described in [28,45] and application of these ideas to Stokes flow can be found in [29,30]. Extension to compatible isogeometric methods can be found in [12] and the connection of isogeometric methods with chains and cochains can be found in [21]. The relation between the Hodge matrix and mass matrices in finite element methods is discussed by Hiptmair and Tarhasaari et al., [22,49].

The outline of this paper is as follows: In Section 2 we introduce the necessary background on vector fields and differential forms. In Section 3 the distinction between forms and pseudo-forms is discussed. In Section 4 a discrete representation of integrals is given and the analogy with the continuous forms in Section 2 is presented. In Section 5 we describe how we can convert continuous forms into discrete forms and vice versa. In Section 6 we demonstrate how the single grid and the dual grid approach can be used to solve the Poisson equation for volume forms. This shows how the method works in practice and also shows that both solutions are equivalent. A straightforward extension to anisotropic diffusivity tensors will be presented also. In Section 7 a brief summary of the paper will be given.

2. Forms and vectors

This section provides an introduction to the basic elements of differential geometry. For more detailed definitions the reader is referred to [18,19]. We emphasize here the distinction between vectors and covectors. Just like the vector field is an extension of the vector concept to manifolds, the 1-form is an extension of the covector to manifolds.

As Burke [13] puts it:

Were this a mere change in notation, it would make no sense to change things. It is not a mere change in notation, however, but a basic change in the fundamental concepts. The new concepts are better for unarguable reasons: they [differential forms] correctly represent a larger symmetry group, and therefore correctly represent more features of the real world.

2.1. Tangent vectors and vector fields

Before introducing differential forms, we need to define vectors or – more precisely – *tangent vectors* in a domain $\mathcal{M}$. In general, $\mathcal{M}$ is a differentiable manifold. Let $\gamma(t)$ be a smooth curve in $\mathcal{M}$ parametrized by $t \in (-\epsilon, \epsilon)$, $\epsilon > 0$, with