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Boundary treatment for fourth-order staggered mesh discretizations of the incompressible Navier–Stokes equations

B. Sanderse [a,b,*], R.W.C.P. Verstappen [c], B. Koren [b,d]

[a] Energy Research Centre of the Netherlands (ECN), The Netherlands
[b] Centrum Wiskunde & Informatica (CWI), The Netherlands
[c] Institute for Mathematics and Computing Science, University of Groningen, The Netherlands
[d] Department of Mathematics and Computer Science, Eindhoven University of Technology, The Netherlands

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A B S T R A C T

Harlow and Welch [Phys. Fluids 8 (1965) 2182–2189] introduced a discretization method for the incompressible Navier–Stokes equations conserving the secondary quantities kinetic energy and vorticity, besides the primary quantities mass and momentum. This method was extended to fourth order accuracy by several researchers [25,14,21]. In this paper we propose a new consistent boundary treatment for this method, which is such that continuous integration-by-parts identities (including boundary contributions) are mimicked in a discrete sense. In this way kinetic energy is exactly conserved even in case of non-zero tangential boundary conditions. We show that requiring energy conservation at the boundary conflicts with order of accuracy conditions, and that the global accuracy of the fourth order method is limited to second order in the presence of boundaries. We indicate how non-uniform grids can be employed to obtain full fourth order accuracy.

1. Introduction

Recently, *symmetry-preserving* and *energy-conserving* discretization methods have gained popularity for the solution of the incompressible Navier–Stokes equations [26,18]. In a symmetry-preserving discretization, the discrete difference operators mimic certain symmetry properties of the continuous differential operators. These discrete symmetry properties ensure that not only mass and momentum are conserved, but also derived quantities such as kinetic energy. Conservation of kinetic energy implies unconditional (non-linear) stability and absence of numerical diffusion. The latter is a favorable property when simulating turbulent flows with LES and DNS [9,13,17].

When combining symmetry preservation with high-order accuracy one can obtain very efficient methods for simulating turbulence [26]. Examples of high-order symmetry-preserving methods on Cartesian grids are the following. On uniform grids, Morinishi et al. [14] constructed a fourth order finite difference method which conserves mass, momentum and kinetic energy. On non-uniform grids strict conservation and a fourth order *local* truncation error could not be obtained. Vasilyev [21] uses the idea of symmetry preservation to extend the finite difference method of Morinishi et al. [10] to non-uniform grids, but he is not able to obtain simultaneous conservation of mass, momentum and energy on non-uniform grids (not even for the classic second-order method of Harlow and Welch). Ham et al. [5] show how the interpolation of the convective velocities should be performed on non-uniform meshes. They argue that second order is recovered for smoothly varying meshes. Verstappen and Veldman [25,26] recognized that on non-uniform grids conservation could be obtained by respecting the symmetries of the underlying operators and keeping the weights of the difference operators mesh

* Corresponding author at: Energy research Centre of the Netherlands (ECN), The Netherlands.
E-mail addresses: bsanderse@gmail.com (B. Sanderse), r.w.c.p.verstappen@rug.nl (R.W.C.P. Verstappen), b.koren@tue.nl (B. Koren).

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independent. Their method (which is the finite-volume equivalent of [14] on uniform grids) conserves mass, momentum and kinetic energy on both uniform and non-uniform grids. Their results show that the local truncation error, on which [14,21,5] focus, does *not* have to be fourth order to let the *global* error be fourth order. This result is in agreement with theoretical work of Manteuffel and White [10].

An ongoing challenge in combining symmetry properties with high order is the prescription of discrete boundary conditions. Morinishi et al. [14] introduce ghost cells with values obtained by higher-order extrapolation and enforcing momentum conservation, but they do not respect the skew-symmetry of the convective operator. Furthermore pressure boundary conditions are prescribed, something which is not necessary in the original method of Harlow and Welch. Desjardins et al. [3] extend the approach of [14] to higher order. Their method is not energy-conserving for orders higher than two; the *higher* the order of the method, the *larger* the energy-conservation error. Again, explicit boundary conditions for the pressure are prescribed. The boundary conditions given in Verstappen and Veldman [26] are energy conserving for homogeneous Dirichlet conditions, but do not mimic the continuous integral properties in a discrete sense if the boundary conditions are non-homogeneous. Furthermore these conditions effectively introduce a Dirichlet condition for the pressure.

In this paper we propose new boundary conditions for the fourth order method such that mass, momentum and kinetic energy are conserved on a discrete level, including the case of non-homogeneous boundary conditions. First, in Section 2 we derive the conservation properties of the continuous equations from a number of integral identities. In Section 3 these integral identities are enforced in a discrete setting, for both the second and fourth order scheme, by a proper choice of boundary volumes and boundary conditions. In Section 4 we investigate the effect of the boundary treatment on the local and global error by employing a one-dimensional error analysis. This theoretical accuracy analysis is supported by numerical experiments in Section 5. In this section we also show the conservation properties of the newly proposed boundary conditions.

2. Conservation properties of the continuous equations

In this section we will discuss a number of properties of the continuous incompressible Navier–Stokes equations: conservation of mass, momentum and kinetic energy, and the equation for the pressure. These properties will be mimicked in a discrete sense in Section 3.

2.1. Mass and momentum

We consider the incompressible Navier–Stokes equations, written as

$$\nabla\cdot\boldsymbol{u}=0, \tag{1}$$

$$\frac{\partial\boldsymbol{u}}{\partial t}+\nabla\cdot(\boldsymbol{u}\boldsymbol{u})=-\nabla p+\nu\nabla^2\boldsymbol{u}, \tag{2}$$

in a domain Ω with boundary $\Gamma=\partial\Omega$, and supplemented with either no-slip boundary conditions

$$\boldsymbol{u}=\boldsymbol{u}_b \quad \text{on } \Gamma \tag{3}$$

or periodic boundary conditions, and a divergence-free initial condition

$$\boldsymbol{u}=\boldsymbol{u}_0. \tag{4}$$

See for example Gresho and Sani [4] for well-posedness of these equations. In integral form these equations read:

$$\int\limits_{\partial\Omega}\boldsymbol{u}\cdot\boldsymbol{n}\,\mathrm{d}\Gamma=0, \tag{5}$$

$$\int\limits_{\Omega}\frac{\partial\boldsymbol{u}}{\partial t}\,\mathrm{d}\Omega+\int\limits_{\partial\Omega}\boldsymbol{u}\boldsymbol{u}\cdot\boldsymbol{n}\,\mathrm{d}\Gamma=-\int\limits_{\partial\Omega}p\boldsymbol{n}\,\mathrm{d}\Gamma+\int\limits_{\partial\Omega}\nu\nabla\boldsymbol{u}\cdot\boldsymbol{n}\,\mathrm{d}\Gamma, \tag{6}$$

expressing conservation of mass and momentum in the domain Ω. In case of periodic boundary conditions, all boundary integrals disappear and one obtains the global momentum balance

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\Omega}\boldsymbol{u}\,\mathrm{d}\Omega=0. \tag{7}$$

In case the normal velocity vanishes at the boundary (no penetration, $\boldsymbol{u}\cdot\boldsymbol{n}=0$), only the contribution of the convective terms disappears.