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A finite element exterior calculus framework for the rotating shallow-water equations

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ABSTRACT

We describe discretisations of the shallow-water equations on the sphere using the framework of finite element exterior calculus, which are extensions of the mimetic finite difference framework presented in Ringler (2010) [11]. The exterior calculus notation provides a guide to which finite element spaces should be used for which physical variables, and unifies a number of desirable properties. We present two formulations: a "primal" formulation in which the finite element spaces are defined on a single mesh, and a "primal–dual" formulation in which finite element spaces on a dual mesh are also used. Both formulations have velocity and layer depth as prognostic variables, but the exterior calculus framework leads to a conserved diagnostic potential vorticity. In both formulations we show how to construct discretisations that have mass-consistent (constant potential vorticity stays constant), stable and oscillation-free potential vorticity advection.

1. Introduction

In a recent paper on horizontal grids for global weather and climate models, [1] listed a number of desirable properties that a numerical discretisation should have, which can be paraphrased as accurate representation of geostrophic adjustment, mass conservation, curl-free pressure gradient, energy-conserving pressure terms, energy-conserving Coriolis term, steady geostrophic modes, and absence/control of spurious modes. Of this list as presented here, the first property could be said to relate to the stability and accuracy of the discrete Laplacian formed from divergence and gradient operators, whilst the next five all relate to mimetic properties (*i.e.* the numerical discretisations exactly represent differential calculus identities such as $\nabla \times \nabla = 0$), and the last property relates to the kernels of the various discretised operators (see [2–4] and related papers by Le Roux and coworkers for extended discussion of these issues in the context of finite element methods). In the context of the rotating shallow-water equations on the sphere, which represent the standard nonlinear framework for investigating horizontal grids for global models, the C-grid staggering on the latitude–longitude grid combined with an appropriate choice of reconstruction of the Coriolis term provides all of these properties, but leaves us with a grid system with a polar singularity. This, together with a need for models with variable resolution, has started a quest for alternative grids and discretisations that satisfy these properties.

The extension of the C-grid to triangular meshes (and the finite element analogue, the RT0-P0 discretisation) satisfies the first six properties and has been popular in both atmosphere and ocean applications [5,6], however it is now well understood that the triangular C-grid supports spurious inertia–gravity mode branches because of the decreased ratio of velocity degrees of freedom (DOFs) to pressure DOFs relative to quadrilaterals (from 2:1 to 3:2) [7,8]. More recently, a Coriolis reconstruction for the hexagonal C-grid was derived in [9] that provides the mimetic properties described above, and this was extended to arbitrary orthogonal C-grids (grids in which dual grid edges that join pressure points intersect the primal grid

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edges orthogonally) in [10]. The hexagonal C-grid has an increased ratio of velocity DOFs to pressure DOFs (from 2:1 to 3:1), and so does not support spurious inertia–gravity mode branches, but does have a branch of spurious Rossby modes. This reconstruction can be used to construct energy and enstrophy conserving C-grid discretisations for the nonlinear rotating shallow-water equations using the vector invariant form [11], in which mimetic properties are used to produce a velocity–pressure formulation in which the diagnosed potential vorticity is locally conserved in a shape-preserving advection scheme, and is consistent with the discrete mass conservation (*i.e.* constant potential vorticity stays constant in the unforced case).

Two directions remain outstanding from this approach, namely the relaxation of the orthogonality requirement which constrains cubed sphere grids so that grid resolution increases much more quickly in the corners than at the middle of the faces [12], and the construction of higher-order operators to avoid grid imprinting. Two recent papers by the authors attempted to address these issues. In [13] a framework was set up to generalise the mimetic approach of [11] to non-orthogonal grids, but the method of constructing sufficiently high-order operators was not clear. Meanwhile, in [14], it was shown that mixed finite element methods in the framework of finite element exterior calculus (see [15] for a review) provide the first six properties listed above, plus sufficient flexibility to adjust the ratio of velocity DOFs to pressure DOFs to 2:1 to avoid spurious mode branches. The BDFM1 space on triangles and the RTk hierarchy of spaces on quadrilaterals were advocated as examples of spaces that satisfy that ratio. However, in that paper it was not clear how the extension to nonlinear shallow-water equations would be made. In this paper we address both of these open questions by describing a finite element exterior calculus framework for the shallow-water equations, which enables us to write the equations in a very compact form that is coordinate-free, and reveals the underlying structure behind the mimetic properties. The goal is to have a numerical discretisation for the shallow-water equations with velocity and layer thickness as prognostic variables, but with a conserved diagnostic potential vorticity. We shall discuss two formulations: a primal grid formulation in which potential vorticity is represented on a continuous finite element space, and a primal–dual grid formulation that makes use of the discrete Hodge star operator introduced in [16,17] in which potential vorticity is represented on a discontinuous finite element space. In the latter case, discontinuous Galerkin or finite volume methods can be used for locally conservative, bounded, mass-consistent potential vorticity advection, whilst in the former case we show that streamline-upwind Petrov–Galerkin methods with discontinuity-capturing schemes can be incorporated into the framework to provide conservative, high-order, stable, non-oscillatory advection of potential vorticity.

Throughout the paper we express our formulations in the language of differential forms. In [15,18] it was shown that this language provides a unifying structure for a wide range of different finite element spaces, which provides a coherent framework for finite element approximation theory and stability theory where previously there was a broad range of bespoke techniques of proof for specific cases. This framework has yielded new finite element spaces and new stability proofs. In this paper, we make use of this framework to design new numerical schemes for the rotating shallow-water equations. The approach makes clear what kind of geometric objects are being dealt with in the equations, and whether they should be interpreted as point values, edge integrals, or cell integrals. In particular, the approach makes it clear which terms involve the metric (and are necessarily more complicated, especially on unstructured grids), and which do not (and hence should be easy to discretise in a simple and efficient way). Furthermore, the exterior derivative d is a very simple operation, since it requires no metric information; this should be reflected by choosing a simple discrete form of d. The fundamental reason why curl-grad and div-curl both vanish is because $\mathrm{d}^2 = 0$; fundamentally these are very simple properties and this should be reflected in the discretisation.

The rest of this paper is structured as follows. In Section 2 we provide a “hands-on” introduction to the calculus of differential forms, then write the rotating shallow-water equations in differential form notation. In Section 3.2 we describe our primal grid finite element exterior calculus formulation of the shallow-water equations, and in Section 3.3 we describe our primal/dual grid formulation. In Section 4, we present some numerical results obtained using these methods. Finally, in Section 5, we provide a summary and outlook.

2. Differential forms on manifolds

In this section, we introduce the required concepts from the language of differential forms, in an informal manner where we shall quote a number of basic results without proof. For more rigorous definitions, the reader is referred to [19,15,20]. We then combine these concepts to write the rotating shallow-water equations on the sphere in differential form notation.

2.1. Differential form preliminaries

Solution domain We shall consider the case in which the solution domain Ω is a closed compact oriented two-dimensional surface. In applications the main surfaces of interest are the surface of the sphere, or a rectangle in the x–y plane with periodic boundary conditions in both Cartesian directions. For brevity of notation we do not consider domains with boundaries; this avoids the need to include boundary terms when integrating by parts, although they can easily be included.

It is useful to define local coordinates on a patch $U \subset \Omega$ *via* an invertible mapping $\phi_U : U \to V \subset \mathbb{R}^2$; the coordinates of a point $\boldsymbol{x} \in U$ are given by the value of $(x^1, x^2) = \phi_U(\boldsymbol{x})$.

Vector fields The tangent space $T_{\boldsymbol{x}}\Omega$ associated with a point $\boldsymbol{x} \in \Omega$ is the space of vectors that are tangent to Ω at $\boldsymbol{x}$. A vector field $\boldsymbol{u}$ on Ω is a mapping from each point $\boldsymbol{x} \in \Omega$ to the tangent space $T_{\boldsymbol{x}}\Omega$, *i.e.* it is a velocity field that is everywhere tangent to Ω. We denote $\mathfrak{X}(\Omega)$ as the space of vector fields on Ω. On a coordinate patch U with coordinates (x^1, x^2), a vector field $\boldsymbol{u}$ can be expanded in the basis $(\partial/\partial x^1, \partial/\partial x^2)$ as $\sum_{i=1}^{2} u^i \frac{\partial}{\partial x^i}$.