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A Correction Function Method for Poisson problems with interface jump conditions

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ABSTRACT

In this paper we present a method to treat interface jump conditions for constant coefficients Poisson problems that allows the use of standard "black box" solvers, without compromising accuracy. The basic idea of the new approach is similar to the Ghost Fluid Method (GFM). The GFM relies on corrections applied on nodes located across the interface for discretization stencils that straddle the interface. If the corrections are solution-independent, they can be moved to the right-hand-side (RHS) of the equations, producing a problem with the same linear system as if there were no jumps, only with a different RHS. However, achieving high accuracy is very hard (if not impossible) with the "standard" approaches used to compute the GFM correction terms.

In this paper we generalize the GFM correction terms to a correction function, defined on a band around the interface. This function is then shown to be characterized as the solution to a PDE, with appropriate boundary conditions. This PDE can, in principle, be solved to any desired order of accuracy. As an example, we apply this new method to devise a 4th order accurate scheme for the constant coefficients Poisson equation with discontinuities in 2D. This scheme is based on (i) the standard 9-point stencil discretization of the Poisson equation, (ii) a representation of the correction function in terms of bicubics, and (iii) a solution of the correction function PDE by a least squares minimization. Several applications of the method are presented to illustrate its robustness dealing with a variety of interface geometries, its capability to capture sharp discontinuities, and its high convergence rate.

1. Introduction

1.1. Motivation and background information

In this paper we present a new and efficient method to solve the constant coefficients Poisson equation in the presence of discontinuities across an interface, with a high order of accuracy. Solutions of the Poisson equation with discontinuities are of fundamental importance in the description of fluid flows separated by interfaces (*e.g.* the contact surfaces for immiscible multiphase fluids, or fluids separated by a membrane) and other multiphase diffusion phenomena. Over the last three decades, several methods have been developed to solve problems of this type numerically [1–17]. However, obtaining a high order of accuracy still poses great challenges in terms of complexity and computational efficiency.

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When the solution is known to be smooth, it is easy to obtain highly accurate finite-difference discretizations of the Poisson equation on a regular grid. Furthermore, these discretizations commonly yield symmetric and banded linear systems, which can be inverted efficiently [18]. On the other hand, when singularities occur (*e.g.* discontinuities) across internal interfaces, some of the regular discretization stencils will straddle the interface, which renders the whole procedure invalid.

Several strategies have been proposed to tackle this issue. Peskin [1] introduced the Immersed Boundary Method (IBM) [1,10], in which the discontinuities are re-interpreted as additional (singular) source terms concentrated on the interface. These singular terms are then "regularized" and appropriately spread out over the regular grid—in a "thin" band enclosing the interface. The result is a first order scheme that smears discontinuities. In order to avoid this smearing of the interface information, LeVeque and Li [3] developed the Immersed Interface Method (IIM) [3,4,11,13], which is a methodology to modify the discretization stencils, taking into consideration the discontinuities at their actual locations. The IIM guarantees second order accuracy and sharp discontinuities, but at the cost of added discretization complexity and loss of symmetry.

The new method advanced in this paper builds on the ideas introduced by the Ghost Fluid Method (GFM) [6–9,12,14,19]. The GFM is based on defining both actual and "ghost" fluid variables at every node on a narrow band enclosing the interface. The ghost variables work as extensions of the actual variables across the interface—the solution on each side of the interface is assumed to have a smooth extension into the other side. This approach allows the use of standard discretizations everywhere in the domain. In most GFM versions, the ghost values are written as the actual values, plus corrections that are independent of the underlying solution to the Poisson problem. Hence, the corrections can be pre-computed, and moved into the source term for the equation. In this fashion the GFM yields the same linear system as the one produced by the problem without an interface, except for changes in the right-hand-side (sources) only, which can then be inverted just as efficiently.

The key difficulty in the GFM is the calculation of the correction terms, since the overall accuracy of the scheme depends heavily on the quality of the assigned ghost values. In [6–9,12] the authors develop first order accurate approaches to deal with discontinuities. In the present work, we show that for the constant coefficients Poisson equation we can generalize the GFM correction term (at each ghost point) concept to that of a correction function defined on a narrow band enclosing the interface. Hence we call this new approach the *Correction Function Method* (CFM). This correction function is then shown to be characterized as the solution to a PDE, with appropriate boundary conditions on the interface—see Section 4. Thus, at least in principle, one can calculate the correction function to any order of accuracy, by designing algorithms to solve the PDE that defines it. In this paper we present examples of 2nd and 4th order accurate schemes (to solve the constant coefficients Poisson equation, with discontinuities across interfaces, in 2D) developed using this general framework.

A key point (see Section 5) in the scheme developed here is the way we solve the PDE defining the correction function. This PDE is solved in a weak fashion using a least squares minimization procedure. This provides a flexible approach that allows the development of a robust scheme that can deal with the geometrical complications of the placement of the regular grid stencils relative to the interface. Furthermore, this approach is easy to generalize to 3D, or to higher orders of accuracy.

1.2. Other related work

It is relevant to note other developments to solve the Poisson equation under similar circumstances—multiple phases separated by interfaces—but with different interface conditions. The Poisson problem with Dirichlet boundary conditions, on an irregular boundary embedded in a regular grid, has been solved to second order of accuracy using fast Poisson solvers [19], finite volume [5], and finite differences [20–23] approaches. In particular, Gibou et al. [21] and Jomaa and Macaskill [22] have shown that it is possible to obtain symmetric discretizations of the embedded Dirichlet problem, up to second order of accuracy. Gibou and Fedkiw [23] have developed a fourth order accurate discretization of the problem, at the cost of giving up symmetry. More recently, the same problem has also been solved, to second order of accuracy, in non-graded adaptive Cartesian grids by Chen et al. [24]. Furthermore, the embedded Dirichlet problem is closely related to the Stefan problem modeling dendritic growth, as described in [25,26].

The finite-element community has also made significant progress in incorporating the IIM and similar techniques to solve the Poisson equation using embedded grids. In particular the works by Gong et al. [15], Dolbow and Harari [16], and Bedrossian et al. [17] describe second order accurate finite-element discretizations that result in symmetric linear systems. Moreover, in these works the interface (or boundary) conditions are imposed in a weak fashion, which bears some conceptual similarities with the CFM presented here, although the execution is rather different.

1.3. Interface representation

Another issue of primary importance to multiphase problems is the representation of the interface (and its tracking in unsteady cases). Some authors (see [1,19,20]) choose to represent the interface explicitly, by tracking interface particles. The location of the neighboring particles is then used to produce local interpolations (*e.g.* splines), which are then applied to compute geometric information—such as curvature and normal directions. Although this approach can be quite accurate, it requires special treatment when the interface undergoes either large deformations or topological changes—such as mergers or splits. Even though we are not concerned with these issues in this paper, we elected to adopt an implicit representation, to avoid complications in future applications. In an implicit representation, the interface is given as the zero level of a function that is defined everywhere in the regular grid—the level set function [27]. In particular, we adopted the Gradient-Augmented Level Set (GA-LS) method [28]. With this extension of the level set method, we can obtain highly accurate