Journal of Computational Physics 230 (2011) 3015–3034

Contents lists available at ScienceDirect

Journal of Computational Physics

journal homepage: www.elsevier.com/locate/jcp

A non-adapted sparse approximation of PDEs with stochastic inputs ☆

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ARTICLE INFO

Article history:
Received 10 June 2010
Received in revised form 24 October 2010
Accepted 4 January 2011
Available online 9 January 2011

Keywords:
Polynomial chaos
Uncertainty quantification
Stochastic PDE
Compressive sampling
Sparse approximation

ABSTRACT

We propose a method for the approximation of solutions of PDEs with stochastic coefficients based on the direct, i.e., non-adapted, sampling of solutions. This sampling can be done by using any legacy code for the deterministic problem as a black box. The method converges in probability (with probabilistic error bounds) as a consequence of sparsity and a concentration of measure phenomenon on the empirical correlation between samples. We show that the method is well suited for truly high-dimensional problems.

1. Introduction

Realistic analysis and design of complex engineering systems require not only a fine understanding and modeling of the underlying physics and their interactions, but also a significant recognition of intrinsic uncertainties and their influences on the quantities of interest. Uncertainty Quantification (UQ) is an emerging discipline that aims at addressing the latter issue; it aims at meaningful characterization of uncertainties in the physical models from the available measurements and efficient propagation of these uncertainties for a quantitative validation of model predictions.

Despite recent growing interests in UQ of complex systems, it remains a grand challenge to efficiently propagate uncertainties through systems characterized by a large number of uncertain sources where the so-called curse-of-dimensionality is yet an unsolved problem. Additionally, development of *non-intrusive* uncertainty propagation techniques is of essence as the analysis of complex multi-disciplinary systems often requires the use of sophisticated coupled deterministic solvers in which one cannot readily intrude to set up the necessary propagation infrastructure.

Sampling methods such as the Monte Carlo simulation and its several variants had been utilized for a long time as the primary scheme for uncertainty propagation. However, it is well understood that these methods are generally inefficient for large-scale systems due to their slow rate of convergence. There has been an increasing recent interest in developing alternative numerical methods that are more efficient than the Monte Carlo techniques. Most notably, the stochastic Galerkin schemes using Polynomial Chaos (PC) bases [38,28,67,2,63] have been successfully applied to a variety of engineering problems and are extremely useful when the number of uncertain parameters is not large. In their original form, the stochastic Galerkin schemes are *intrusive*, as one has to modify the deterministic solvers for their implementation. *Stochastic collocation* schemes [58,45,66,1,51] belong to a different class of methods that rely upon (isotropic) sparse grid integration/interpolation in the stochastic space of the problem to reduce the curse-of-dimensionality associated with the conventional

☆ A preliminary version of this work has appeared in a 2009 *CTR Annual Research Brief* report [35].

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0021-9991/$ - see front matter
doi:10.1016/j.jcp.2011.01.002

tensor-product integration/interpolation rules. As their construction is primarily based on the input parameter space, the computational cost of both stochastic Galerkin and collocation techniques increases rapidly for large number of independent input uncertainties.

More recently, efforts have been made to construct *solution-adaptive* uncertainty propagation techniques that exploit any structures in the solution to decrease the computational cost. Among them are the multi-scale model reduction of [33] and the sparse decomposition of [60,8,6,7] for the stochastic Galerkin technique, adaptive sparse regression approaches of [10,11] for PC expansions, anisotropic and adaptive sparse grids of [50,44] for the stochastic collocation scheme, and the low-rank solution approximations of [52,53,34].

In the present study, we are interested in cases where the quantity of interest is *sparse* at the stochastic level, i.e., it can be accurately represented with only *few* terms when linearly expanded into a stochastic, e.g., polynomial chaos, basis. Interestingly, sparsity is salient in the analysis of high-dimensional problems where the number of energetic basis functions (those with large coefficients) is small relative to the cardinality of the full basis. For instance, it has been shown in [60,8] that, under some mild conditions, solutions to linear elliptic stochastic PDEs with high-dimensional random coefficients admit sparse representations with respect to the PC basis. Consequently, an approach based on a zero-dimensional algebraic stochastic problem has been proposed in [8] to detect the sparsity pattern, which then guides the stochastic Galerkin analysis of the original problem. Moreover, a "quasi"-best N-term approximation for a class of elliptic stochastic PDEs has been proposed in [7].

In this work, using *concentration of measure* inequalities and *compressive sampling* techniques, we derive a method for PC expansion of sparse solutions to stochastic PDEs. The proposed method is

- Non-intrusive: it is based on the direct random sampling of the PDE solutions. This sampling can be done by using any legacy code for the deterministic problem as a black box.
- Non-adapted: it does not tailor the sampling process to identify the important dimensions at the stochastic level.
- Provably convergent: we obtain probabilistic bounds on the approximation error proving the stability and convergence of the method.
- Well-suited to problems with high-dimensional random inputs.

Compressive sampling is an emerging direction in signal processing that aims at recovering *sparse* signals accurately (or even exactly) from a small number of their random projections [20,21,17,30,15,18,16,22,14]. A sparse signal is simply a signal that has only few significant coefficients when linearly expanded into a basis, e.g., $\{\psi_{\boldsymbol{\alpha}}\}$.

For sufficiently sparse signals, the number of samples needed for a successful recovery is typically less than what is required by the Shannon–Nyquist sampling principle. Generally speaking, a successful signal reconstruction by compressive sampling is conditioned upon:

- *Sufficient* sparsity of the signal; and
- *Incoherent* random projections of the signal.

A square-measurable stochastic function $u(\omega)$, defined on a suitable probability space $(\Omega,\mathcal{F},\mathcal{P})$ can be expanded into a mean-squared convergent series of the chaos polynomial bases, i.e., $u(\omega)\approx\sum_{\boldsymbol{\alpha}}c_{\boldsymbol{\alpha}}\psi_{\boldsymbol{\alpha}}(\omega)$, with some cardinality P. The stochastic function $u(\omega)$ is then sparse in PC basis $\{\psi_{\boldsymbol{\alpha}}\}$, if only a small fraction of coefficients $c_{\boldsymbol{\alpha}}$ are significant. In this case, under certain conditions, the sparse PC coefficients $\boldsymbol{c}$ may be computed accurately and robustly using only $N \ll P$ random realizations of $u(\omega)$ via compressive sampling. Given N random samples of $u(\omega)$, compressive sampling aims at finding the sparsest (or nearly sparsest) coefficients $\boldsymbol{c}$ from an optimization problem of the form

$$(P_{s,\delta}) : \min_{\boldsymbol{c}} \|\boldsymbol{W}\boldsymbol{c}\|_s \quad \text{subject to} \quad \|\boldsymbol{\Psi}\boldsymbol{c}-\boldsymbol{u}\|_2 \leqslant \delta, \tag{1}$$

where $\|\boldsymbol{W}\boldsymbol{c}\|_s$, with $s = \{0,1\}$ and some positive diagonal weight matrix $\boldsymbol{W}$, is a measure of the sparsity of $\boldsymbol{c}$ and $\|\boldsymbol{\Psi}\boldsymbol{c}-\boldsymbol{u}\|_2$ is a measure of the accuracy of the truncated PC expansion in estimating the $u(\omega)$ samples. The N-vector $\boldsymbol{u}$ contains the independent random samples of $u(\omega)$ and the rows of the $N\times P$ matrix $\boldsymbol{\Psi}$ consist of the corresponding samples of the PC basis $\{\psi_{\boldsymbol{\alpha}}\}$.

Throughout the rest of this manuscript, we will elaborate on the formulation of the compressive sampling problem (1) and the required conditions under which it leads to an accurate and stable approximation of an arbitrary sparse stochastic function as well as sparse solutions to linear elliptic stochastic PDEs. Although we choose to study this particular class of stochastic PDEs, we stress that the proposed algorithms and theoretical developments are far more general and may be readily applied to recover sparse solution of other stochastic systems.

In Section 2, we describe the setup of the problem of interest for which numerical experiments are performed. We then, in Section 3, lay out the main contributions of the present work. In particular, the approximation of sparse stochastic functions as well as the elliptic stochastic PDEs using the compressive sampling technique are introduced in Sections 3.2–3.4. Sections 3.5 and 3.6 discuss some of the implementation details of the present technique. To demonstrate the accuracy and efficiency of the proposed procedures, in Section 4, we perform two numerical experiments on 1-*D* (in space) linear elliptic stochastic differential equations with high-dimensional random diffusion coefficients.