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# Artificial boundary conditions for certain evolution PDEs with cubic nonlinearity for non-compactly supported initial data

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ABSTRACT

The paper addresses the problem of constructing non-reflecting boundary conditions for two types of one dimensional evolution equations, namely, the cubic nonlinear Schrödinger (NLS) equation, $\partial_t u + \mathcal{L}u - i\chi|u|^2 u = 0$ with $\mathcal{L} \equiv -i\partial_x^2$, and the equation obtained by letting $\mathcal{L} \equiv \partial_x^3$. The usual restriction of compact support of the initial data is relaxed by allowing it to have a constant amplitude along with a linear phase variation outside a compact domain. We adapt the pseudo-differential approach developed by Antoine et al. (2006) [5] for the NLS equation to the second type of evolution equation, and further, extend the scheme to the aforementioned class of initial data for both of the equations. In addition, we discuss efficient numerical implementation of our scheme and produce the results of several numerical experiments demonstrating its effectiveness.



## 1. Introduction

For evolution problems formulated on an unbounded domain, *transparent boundary conditions* (TBCs) facilitate truncation to a bounded domain for the numerical computation of solutions. If the evolution is in time, such conditions usually appear as a Dirichlet to Neumann map nonlocal in time. While the construction of such exact conditions for a class of initial conditions which have a compact support embedded in the computational domain is possible for linear evolution equations [1,2] and integrable nonlinear equations [3,4], the general problem can only be treated by introducing certain approximate boundary conditions dubbed as *artificial boundary conditions* (ABCs) [5–9]. To have an overview of the state-of-the-art of the subject we refer to the review article [11]. In order to be able to successfully construct TBCs/ABCs it is mandatory to impose certain behavior on the initial condition outside the computational domain. For this reason, in most of the aforementioned works one has the restriction that the initial data must have a compact support in $\mathbb{R}$. In the present work we relax this restriction and consider a rather broader class of initial conditions such that, outside a compact domain, the profile has a constant amplitude with a linear phase variation. Part of the motivation for this effort comes from the fact that certain systems do admit of physically interesting solutions which fall in this class as far as the initial value problem (IVP) is concerned. Therefore, it seems worthwhile to consider such a general class of IVPs not only from the point of view of achieving greater generality on the class of initial data towards construction of TBCs/ABCs but also from the point of view of physical application.

We consider the two following nonlinear evolution equations occurring in nonlinear optics [12]:

$$i\partial_t u + \partial_x^2 u + \chi|u|^2 u = 0, \tag{1}$$

$$\partial_t u + \partial_x^3 u - i\chi|u|^2 u = 0. \tag{2}$$

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The propagation of light pulses in Kerr-type nonlinear media where the dominant dispersion is of the second order is governed by an equation of the type (1), the so called NLS equation, where $\chi$ is a real parameter. At the zero of the second order dispersion, the dominant dispersion is of the third order leading to (2) as the governing equation whose linear operator resembles the linearized Korteweg-de-Vries (KdV) operator. As far as the class of initial conditions is concerned we would like to mention that the NLS equation has, as an outcome of the theory of inverse scattering, several interesting solutions of this class, for instance, a bright soliton solution on a cw background [13] and a dark/gray soliton [14].

In the search for a method to construct Dirichlet-to-Neumann-map-type ABCs for the present class of initial conditions, incidently for the Eq. (2), we found the method developed by Antoine et al. [5] to be a very favorable starting point. The method described by the authors, for the case of compactly supported initial data, is based on quasi-homogeneous pseudo-differential operator calculus as developed by Lascar [15] and proceeds by seeking a Nirenberg-type factorization [17] of the relevant operator. We propose an extension to the earlier scheme by, first, linearizing the nonlinear equation to a nonhomogeneous equation (instead of a homogeneous one as in [5]) in order to accommodate for the lack of compact support of the initial data, and then, constructing certain inverse operators which are otherwise not needed in the homogeneous case to obtain the ABCs. The general version of the ABCs is again obtained under the high frequency assumption on the solution as in [5]. Let us point out that our method could also be applied to the pseudo-differential strategy employed by Szeftel [6] but we do not pursue this programme here. For the case of compactly supported initial data, the gauge change strategy in [5] tends to give better results as compared to the potential strategy of Szeftel for the NLS equation.

As far as the generality of the method presented in this paper (as well as that in [5]) is concerned, at the outset one can conclude that it directly depends on the success of obtaining a Nirenberg-type [17] factorization for the linearized version of the original nonlinear evolution PDE as in [5]. For the case of compactly supported initial data one may succeed [7], however, the lack of compact support presents us with additional difficulties. It follows that the generality of the method now solely depends on the possibility of transforming the original nonlinear evolution PDE into a form (most likely to be nonhomogeneous equation similar to the present case) which allows for a Nirenberg-type factorization in the new setting. The method presented in this paper can then be easily adapted to those systems. However, a word of caution should be added here about the accuracy of the scheme; it is evident that a reasonably accurate scheme must be able to evolve the distinct cw backgrounds, present in the initial data, alone with sufficient accuracy otherwise such cases tend to become inadmissible (fortunately, this problem is not encountered in the present paper as the scheme obtained is exact for cw backgrounds alone).

Since the Eq. (2) occupied our primary interest and to the best of our knowledge has not been the subject matter of study in the present context, we begin with a thorough treatment of the Eq. (2) in Section 2 which is then followed by Section 3 where we adapt the same scheme to the NLS equation. The subsequent sections are devoted to the numerical study and discussion of the results for the proposed scheme.

## 2. Construction of ABCs for $\partial_t u + \partial_x^3 u - i\chi|u|^2 u = 0$

Let us consider the initial value problem defined by

$$\begin{aligned} &\partial_t u + \partial_x^3 u - i\chi|u|^2 u = 0, \quad (x,t) \in \mathbb{R}\times\mathbb{R}_+,\\ &u(x,0) = u_0(x), \end{aligned} \tag{3}$$

where $u_0(x) \in \mathcal{C}^\infty$ such that $u_0(x) = A^{(0)}_{l,r} e^{i\kappa_{l,r}x}$ for $x \in \Omega_{l,r}$, respectively, where $\Omega_r = [x_r, \infty)$ is considered as the right exterior domain and $\Omega_l = (-\infty, x_l]$ the left exterior domain. For the purpose of numerical solution the computation domain is taken to be $\Omega_i = (x_l, x_r)$. We begin with a discussion of the linearized form of the Eq. (3). The construction of TBCs for the linearized case was considered by Zheng et al. [2] for the case of compactly supported initial data. Using the substitution

$$\psi(x,t) = u(x,t) - A^{(0)}_{l,r} e^{i(\kappa_{l,r}x + \kappa^3_{l,r}t)}, \quad x \in \Omega_{l,r} \text{ respectively} \tag{4}$$

and burrowing the results from [2], it easily follows that an equivalent formulation of the initial value problem for linearized form of (3) on the bounded domain $\Omega_i$ is given by

$$\begin{aligned} &\partial_t u + \partial_x^3 u = 0, \quad (x,t) \in \Omega_i \times \mathbb{R}_+,\\ &u(x,0) = u_0(x) \end{aligned} \tag{5}$$

with the boundary conditions:

$$\begin{cases} \partial_x^2\psi + \partial_t^{2/3}\psi - \partial_t^{1/3}\partial_x\psi = 0, \quad x = x_l,\\ \partial_x\psi + \partial_t^{1/3}\psi = 0, \quad \partial_x^2\psi - \partial_t^{2/3}\psi = 0, \quad x = x_r, \end{cases} \tag{6}$$

which serve as the TBCs or the generalized Dirichlet to Neumann map at the boundary needed for the truncation of the infinite domain. The fractional operators appearing in (6) are defined as follows (see e.g. Miller and Ross [16]): For $\alpha > 0$, $\partial_t^{-\alpha}$ denotes the Riemann–Liouville fractional integral given by

$$\partial_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds, \tag{7}$$