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ScienceDirect
Journal of Computational Physics 220 (2006) 249–269
JOURNAL OF COMPUTATIONAL PHYSICS
www.elsevier.com/locate/jcp

High order finite difference methods for wave propagation in discontinuous media

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Received 20 June 2005; received in revised form 8 February 2006; accepted 9 May 2006
Available online 30 June 2006

Abstract

High order finite difference approximations are derived for the second order wave equation with discontinuous coefficients, on rectangular geometries. The discontinuity is treated by splitting the domain at the discontinuities in a multi block fashion. Each sub-domain is discretized with compact second derivative summation by parts operators and the blocks are patched together to a global domain using the projection method. This guarantees a conservative, strictly stable and high order accurate scheme. The analysis is verified by numerical simulations in one and two spatial dimensions.

Keywords: High order finite difference methods; Wave equation; Numerical stability; Second derivatives; Discontinuous media

1. Introduction

For wave propagating problems, the computational domain is often large compared to the wavelengths, which means that waves have to travel long distances during long times. As a result, high order accurate time marching methods, as well as high order spatially accurate schemes (at least third order) are required. Such schemes, although they might be G-K-S stable [11] (convergence to the true solution as $\Delta x \rightarrow 0$),

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0021-9991/$ - see front matter
doi:10.1016/j.jcp.2006.05.007

may exhibit a non-physical growth in time [3], for realistic mesh sizes. It is therefore important to devise schemes that do not allow a growth in time that is not called for by the differential equation. Such schemes are called strictly (or time) stable.

In many applications, like general relativity [29], seismology and acoustics, the underlying equations are systems of second order hyperbolic partial differential equations. However (as pointed out in [15]), with very few exceptions, the equations are rewritten and solved on first order form. There are three obvious drawbacks with this approach, namely (i) we double the number of unknowns, (ii) we might introduce spurious oscillations (due to unresolved features), and (iii) we need twice as many grid points (both in time and in each of the spatial dimensions) to obtain the same accuracy. The reasons for solving the equations on first order form are probably due to the fact that computational methods for first order hyperbolic systems are very well developed, and they are naturally more suited for complex geometries.

For acoustic and electromagnetic wave propagation, staggered grid discretizations are very popular [6,31] since that avoids (ii) and (iii) above. Note however (again see [15]) that staggering in both time and space is more or less equivalent to solving the system of equations on second order form. One major disadvantage is that staggered grids do not have the summation by parts (SBP) property and that can lead to complications at boundaries and internal interfaces, especially for high order discretizations. To retain high order accuracy for problems with discontinuities in the coefficients is another concern [12,13,7].

The methods discussed above all solve the equations on first order form. Difference approximations have previously been derived [15,16,25,1,5] for the second order wave equation, without first writing it as a first order system. For problems with discontinuous coefficients at most second order accuracy have been recovered [1,5,13].

The second derivative terms have received little attention, especially concerning the stability issues for high order approximations [2]. Finite difference operators approximating second derivatives and satisfying a summation by parts rule, have previously been derived [20] for the 4th, 6th and 8th order case, with the emphasis on strictly stable formulations to mixed hyperbolic–parabolic problems.

One major advantage of using SBP operators [17,18,27] to discretize the equations on a multi block domain is that we can mimic the boundary and interface terms from the underlying continuous problem. Given the continuous boundary and interface conditions (i.e., the physics) in combination with the simultaneous approximation term (SAT) method [3,4,21,22] or the projection method [23,24] we can obtain completely analogous conservation and stability properties as for the underlying partial differential equation (PDE). This should attract physicists to employ this technique for a range of applications. In general relativity for example, the SBP operators combined with the SAT technique have now been successfully implemented [8,19] for system of equations on first order form (in time).

In this paper we will show how a certain class of the recently developed compact and high order accurate second derivative SBP operators [20] can be combined with the projection method for implementing general boundary and interface conditions. On piecewise rectangular domains we show that this technique leads to strictly stable and high order accurate schemes for the wave equation on second order form and discontinuous media. We will also show that the projection method requires special treatment at corners and block interfaces in two dimensions.

We focus on geometrically relative simple problems with piecewise constant coefficients and aim for high accuracy. Typical applications where this technique is appropriate include long range underwater acoustics (layers of air, water and soil), various seismological problems (layers of rock, water and possibly oil) as well as electromagnetic problems (wave guides and printed circuit boards). Complex geometries, varying coefficients and also the problem with absorbing boundary conditions [30,14] will not be addressed in this paper.

In Section 2 we introduce some definitions and discuss the SBP property for the second derivative. In Section 3 we consider the second order wave equation in one dimension (1-D) and show how the projection method and the SBP operators can be combined to obtain strictly stable schemes for problems with discontinuous coefficients. In Section 4 we consider the two-dimensional (2-D) problem. In Section 5 we describe a compact and explicit high order accurate time marching method that involves only two time levels. In Section 6, computations are done and in 7 conclusions are drawn. The SBP operators used in the computations are presented in Appendix II.