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Mixed Constraint Preconditioners for the iterative solution of FE coupled consolidation equations

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ABSTRACT

The Finite Element (FE) integration of the coupled consolidation equations requires the solution of linear symmetric systems with an indefinite saddle point coefficient matrix. Because of ill-conditioning, the repeated solution in time of the FE equations may be a major computational issue requiring ad hoc preconditioning strategies to guarantee the efficient convergence of Krylov subspace methods. In the present paper a Mixed Constraint Preconditioner (MCP) is developed combining implicit and explicit approximations of the inverse of the structural sub-matrix, with the performance investigated in some representative examples. An upper bound of the eigenvalue distance from unity is theoretically provided in order to give practical indications on how to improve the preconditioner. The MCP is efficiently implemented into a Krylov subspace method with the performance obtained in 2D and 3D examples compared to that of Inexact Constraint Preconditioners and Least Square Logarithm scaled ILUT preconditioners. Two variants of MCP (T-MCP and D-MCP), developed with the aim at reducing the cost of the preconditioner application, are also tested. The results show that the MCP variants constitute a reliable and robust approach for the efficient solution of realistic coupled consolidation FE models, and especially so in severely ill-conditioned problems.

1. Introduction

The time-dependent displacements and fluid pore pressure in porous media are controlled by the consolidation theory. This was first mathematically described by Biot [1], who coupled the elastic equilibrium equations with a continuity or mass balance equation to be solved under appropriate boundary and initial flow and loading conditions.

The coupled consolidation equations are typically solved numerically using Finite Elements (FE) in space, thus giving rise to a system of first-order differential equations the solution to which is addressed by an appropriate time marching scheme. A major computational issue is the repeated solution in time of the resulting discretized indefinite equations, which can be generally written as

$$\mathcal{A}\boldsymbol{x} = \boldsymbol{b}, \quad \text{where } \mathcal{A} = \begin{bmatrix} K & B^{\mathrm{T}} \\ B & -C \end{bmatrix}. \tag{1}$$

Both the sub-matrices K and C are symmetric positive definite (SPD). Denoting with m the number of FE nodes, $C \in \mathbb{R}^{m\times m}$, $B \in \mathbb{R}^{m\times n}$ and $K \in \mathbb{R}^{n\times n}$, where n is equal to $2m$ or $3m$ according to the spatial dimension of the problem if the same interpolation is used for displacement and pressure variables.

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The use of iterative solvers is recommended in large size realistic consolidation models. Among them, projection (or conjugate gradient-like) methods based on Krylov subspaces for indefinite systems, such as BiCGStab (Bi-Conjugate Gradient Stabilized [2]), are attracting a growing interest on the grounds of their robustness and efficiency [3–8]. However, the small time integration steps typically required in the early phase of the analysis may yield a severe ill-conditioning [9], and the selection of an efficient preconditioning strategy turns out to be a key issue to guarantee and accelerate the convergence. Note on passing that popular symmetric Krylov solvers, such as MINRES, cannot be generally used for problem (1) because of the indefiniteness of the preconditioners.

Matrix $\mathcal{A}$ in (1) is a classical example of saddle point problem, which is encountered in other fields as well including constrained optimization, least squares and Navier–Stokes equations. The constraint preconditioners for Krylov solvers in the solution of saddle point problems have been studied by a number of authors [10–16]. In most of the above references the preconditioner is obtained from $\mathcal{A}$ with the (1,1) block K well approximated and replaced by its diagonal. In the coupled consolidation problem, however, K is not diagonally dominant and a better approximation is required to ensure convergence. Bergamaschi et al. [17] have developed both an Exact and an Inexact Constraint Preconditioner (ECP and ICP, respectively) with the explicit approximation of K^{-1} provided by the approximate inverse preconditioner AINV [18]. The ICP variant is suggested with the aim at avoiding the need for exactly solving an inner $m \times m$ linear system for each preconditioner application as is required by ECP. In the present paper a Mixed Constraint Preconditioner (MCP) is developed where an implicit and an explicit approximation of K^{-1} are provided by an incomplete Cholesky decomposition ILLT and AINV, respectively. Using the spectral analysis it is shown that most of the eigenvalues of the preconditioned matrix are real positive and, most importantly, clustered around unity, with the value of the few remaining ones carefully kept under control. Two variants of MCP are then considered, based on the block structure of the preconditioner. The former, called Triangular MCP (T-MCP), uses an upper block triangular approximation of MCP, while the latter, denoted as Diagonal MCP (D-MCP) uses the block diagonal part of MCP only.

The paper is organized as follows. After a brief review of FE coupled consolidation equations, ECP and ICP with their main properties are revisited. In particular, a theoretical bound is given for the ICP eigenspectrum which helps give some practical indications as to the implementation of an effective preconditioner. Then, MCP is developed on the basis of the previous theoretical findings and experimented with in realistic medium and large size 2D and 3D problems. The MCP performance is compared to that of more traditional preconditioning techniques, such as ILUT with optimal fill-in degree [19] and a preliminary Least Square Logarithm (LSL) scaling [6], and that of ICP. The possible use of the T-MCP and D-MCP variants is finally discussed with a few remarks closing the paper.

2. Finite element coupled consolidation equations

The system of partial differential equations governing the 3D coupled consolidation process in fully saturated porous media is derived from the classical Biot's formulation [1] and successive modifications as:

$$(\lambda+\mu)\frac{\partial\epsilon}{\partial i}+\mu\nabla^2 u_i=\alpha\frac{\partial p}{\partial i},\quad i=x,\ y,\ z, \tag{2}$$

$$\frac{1}{\gamma}\nabla(k\nabla p)=[\phi\beta+c_{br}(\alpha-\phi)]\frac{\partial p}{\partial t}+\alpha\frac{\partial\epsilon}{\partial t}, \tag{3}$$

where c_{br} and β are the volumetric compressibility of solid grains and water, respectively, ϕ is the porosity, k the medium hydraulic conductivity, ϵ the medium volumetric dilatation, α the Biot coefficient, λ and μ are the Lamé constant and the shear modulus of the porous medium, respectively, γ is the specific weight of water, ∇ the gradient operator, x, y, z are the coordinate directions, t is time, and p and u_i are the incremental pore pressure and the components of incremental displacement along the i-direction, respectively.

Use of FE in space yields a system of first order differential equations which can be integrated by the Crank–Nicolson scheme [9]. The resulting linear system has to be repeatedly solved to obtain the transient displacements and pore pressures. The unsymmetric matrix controlling the solution scheme reads:

$$A=\begin{bmatrix} K/2 & -Q/2 \\ \frac{Q^{\mathrm{T}}}{\Delta t} & H/2+\frac{P}{\Delta t}\end{bmatrix}, \tag{4}$$

where K, H, P and Q are the elastic stiffness, flow stiffness, flow capacity and flow-stress coupling matrices, respectively. Matrix A can be readily symmetrized by multiplying the upper set of equations by 2 and the lower set by $-\Delta t$, thus obtaining the sparse 2×2 block symmetric indefinite matrix (1) where $B=-Q^{\mathrm{T}}$ and $C=\Delta tH/2+P$.

A major difficulty in the repeated solution to system (1) is the likely ill-conditioning of $\mathcal{A}$ caused by the large difference in magnitude between the coefficients of blocks K, B and C. The generic (i,j) element of each matrix is related to the hydro-mechanical properties of the porous medium as follows [9]:

$$K_{ij}\propto E, \tag{5}$$

$$B_{ij}\propto\sqrt{V}, \tag{6}$$

$$C_{ij}\propto\Delta t\frac{k}{\gamma}+\phi\beta V, \tag{7}$$