Journal of Computational Physics 230 (2011) 2286–2302

Contents lists available at ScienceDirect

Journal of Computational Physics

journal homepage: www.elsevier.com/locate/jcp

Hyperboloidal layers for hyperbolic equations on unbounded domains

Anıl Zenginoğlu *

Laboratoire Jacques-Louis Lions, Université Pierre et Marie Curie (Paris VI), Paris, France
Theoretical Astrophysics, California Institute of Technology, Pasadena, CA, USA

ARTICLE INFO

Article history:
Received 24 August 2010
Received in revised form 11 December 2010
Accepted 13 December 2010
Available online 21 December 2010

Keywords:
Transparent (non reflecting, absorbing) boundary conditions
Perfectly matched layers
Hyperboloidal layers
Hyperboloidal compactification
Wave equations
Maxwell equations

ABSTRACT

We show how to solve hyperbolic equations numerically on unbounded domains by compactification, thereby avoiding the introduction of an artificial outer boundary. The essential ingredient is a suitable transformation of the time coordinate in combination with spatial compactification. We construct a new layer method based on this idea, called the hyperboloidal layer. The method is demonstrated on numerical tests including the one dimensional Maxwell equations using finite differences and the three dimensional wave equation with and without nonlinear source terms using spectral techniques.

1. Introduction

Hyperbolic equations typically admit wavelike solutions that oscillate infinitely many times in an unbounded domain. Take a plane wave in one spatial dimension with frequency ω and wave number k,

$$u(x,t) = e^{2\pi i(kx-\omega t)}. \tag{1}$$

Any mapping of such an oscillatory solution from an infinite domain to a finite domain results in infinitely many oscillations near the domain boundary, which cannot be resolved numerically. We refer to this phenomenon as the compactification problem [1]. It is commonly stated that hyperbolic partial differential equations are not compatible with compactification, and therefore cannot be solved on unbounded domains accurately.

A suitable transformation of the *time coordinate*, however, leads to a finite number of oscillations in an infinite spatial domain. Introduce

$$\tau(x,t) = t - \frac{k}{\omega}\left(x + \frac{C}{1+x}\right), \tag{2}$$

where C is a finite, positive constant. The plane wave (1) becomes

$$u(x,\tau) = e^{-2\pi i(kC/(1+x)+\omega\tau)}. \tag{3}$$

This representation of the plane wave has only kC cycles along a constant time hypersurface in the unbounded space $x \in [0,\infty)$, and is therefore compatible with compactification.

* Address: Theoretical Astrophysics, California Institute of Technology, Pasadena, CA, USA. Tel.: +1 626 354 2514.
E-mail address: anil@caltech.edu

0021-9991/$ - see front matter
doi:10.1016/j.jcp.2010.12.016

The simple idea just described has far reaching consequences. In numerical calculations of hyperbolic equations one typically truncates the unbounded solution domain by introducing an artificial outer boundary that is not part of the original problem. Boundary conditions, called transparent, absorbing, radiative, or nonreflecting; are constructed to simulate transparency of this artificial outer boundary. There has been significant developments in the treatment of artificial outer boundaries since the 70s, but there is no consensus on an optimal method [2,3]. Especially the construction of boundary conditions for nonlinear problems is difficult [4]. A successful technique for numerical calculations on unbounded domains resolves this problem for suitable hyperbolic equations and provides direct quantitative access to asymptotic properties of solutions.

Furthermore, the numerical construction of oscillatory solutions as (3) can be very efficient. Numerical accuracy requirements for hyperbolic equations are typically given in terms of numbers of grid points per wavelength. In the example presented above, the free parameter C determines the number of cycles to be resolved, which may be chosen small. This suggests that high order numerical discretizations requiring a few points per wavelength can be very efficient in combination with time transformations of the type (2).

The rest of the paper is devoted to the discussion of time transformation and compactification for hyperbolic equations. The theoretical part of the paper (Sections 2 and 3) includes a detailed description of the method. We discuss the compactification problem (Section 2.1) and its resolution (Section 2.2) for the advection equation in one dimension. In Section 2.3 we discuss the wave equation with incoming and outgoing characteristics. We show that the method works also for systems of equations (Section 2.4). Hyperboloidal layers are introduced in Section 2.5 in analogy to absorbing layers. In multiple spatial dimensions, compactification is performed in the outgoing direction in combination with rescaling to take care of the asymptotic behavior (Sections 3.1 and 3.2). The layer strategy in multiple dimensions allows us to employ arbitrary coordinates in an inner domain, where sources or scatterers with irregular shapes may be present (Section 3.3). We finish the theoretical part discussing possible generalizations of the method to nonspherical coordinate systems (Section 3.4). Section 4 includes numerical experiments in one and three spatial dimensions. In one dimension, we solve the Maxwell equations using finite difference methods (Section 4.1). A stringent test of the method is the evolution of off-centered initial data for the wave equation in three spatial dimensions with and without nonlinear source terms (Section 4.2). We conclude with a discussion and an outlook in Section 5.

2. Compactification in one spatial dimension

2.1. Spatial compactification

Consider the initial boundary value problem for the advection equation

$$\partial_t u + \partial_x u = 0, \quad u(x,0) = u_0(x), \quad u(0,t) = b(t). \tag{4}$$

The problem is posed on the unbounded domain $x \in [0,\infty)$. We transform the infinite domain in x to a finite domain by introducing the compactifying coordinate ρ via

$$\rho(x) = \frac{x}{1+x}, \quad x(\rho) = \frac{\rho}{1-\rho}. \tag{5}$$

The advection equation becomes

$$\partial_t u + (1-\rho)^2 \partial_\rho u = 0. \tag{6}$$

The spatial domain is now given by $\rho \in [0,1]$. Characteristics of this equation are solutions to the ordinary differential equation

$$\frac{d\rho(t)}{dt} = -(1-\rho(t))^2.$$

They are plotted in Fig. 1. The compactification problem is clearly visible: the coordinate speed of characteristics approaches zero near a neighborhood of the point that corresponds to spatial infinity. The advection equation has a finite speed of propagation, therefore its characteristics can not reach infinity in finite time.

A concrete example illustrates the problem for oscillatory solutions. Set initial data $u_0(x) = \sin(2\pi x)$ and boundary data $b(t) = -\sin(2\pi t)$ in (4). We obtain the solution

$$u(x,t) = \sin(2\pi(x-t)), \tag{7}$$

which reads in the compactifying coordinate (5)

$$u(\rho,t) = \sin\left(2\pi\left(\frac{\rho}{1-\rho} - t\right)\right). \tag{8}$$

The solution is depicted in Fig. 2 at $t = 0$ on $x \in [0,10]$ in the original coordinate and on $\rho \in [0,10/11]$ in the compactifying coordinate. The oscillations can not be resolved in the compactifying coordinate near infinity due to infinite blueshift in spatial frequency.