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On the implementation of WENO schemes for a class of polydisperse sedimentation models

Raimund Bürger [a,*], Rosa Donat [b], Pep Mulet [b], Carlos A. Vega [c]

[a] CI^2MA and Departamento de Ingeniería Matemática, Facultad de Ciencias Físicas y Matemáticas, Universidad de Concepción, Casilla 160-C, Concepción, Chile
[b] Departament de Matemàtica Aplicada, Universitat de València, Av. Dr. Moliner 50, E-46100 Burjassot, Spain
[c] Departamento de Matemáticas y Estadística, División de Ciencias Básicas, Universidad del Norte, Barranquilla, Colombia

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ABSTRACT

The sedimentation of a polydisperse suspension of small rigid spheres of the same density, but which belong to a finite number of species (size classes), can be described by a spatially one-dimensional system of first-order, nonlinear, strongly coupled conservation laws. The unknowns are the volume fractions (concentrations) of each species as functions of depth and time. Typical solutions, e.g. for batch settling in a column, include discontinuities (kinematic shocks) separating areas of different composition. The accurate numerical approximation of these solutions is a challenge since closed-form eigenvalues and eigenvectors of the flux Jacobian are usually not available, and the characteristic fields are neither genuinely nonlinear nor linearly degenerate. However, the flux vectors associated with the widely used models by Masliyah, Lockett and Bassoon (MLB model) and Höfler and Schwarzer (HS model) give rise to Jacobians that are low-rank perturbations of a diagonal matrix. This property allows to apply a convenient hyperbolicity criterion that has become known as the "secular equation" [J. Anderson, A secular equation for the eigenvalues of a diagonal matrix perturbation, Lin. Alg. Appl. 246 (1996) 49–70]. This criterion was recently applied [R. Bürger, R. Donat, P. Mulet, C.A. Vega, Hyperbolicity analysis of polydisperse sedimentation models via a secular equation for the flux Jacobian, SIAM J. Appl. Math. 70 (2010) 2186–2213] to prove that the MLB and HS models are strictly hyperbolic under easily verifiable conditions, that their eigenvalues interlace with the velocities of the species that form the flux vector (so the velocities are good starting values for a root finder), and that the corresponding eigenvectors can be calculated with acceptable effort. In the present work, the newly available characteristic information is exploited for the implementation of characteristic-wise (spectral) weighted essentially non-oscillatory (WENO) schemes for the MLB and HS models. Numerical examples illustrate that WENO schemes which use this spectral information are superior in resolution, and even in efficiency for the same overall resolution, to component-wise WENO schemes.

1. Introduction

1.1. Scope

This work concerns high-resolution numerical schemes for systems of conservation laws that arise as one-dimensional kinematic models for the sedimentation of polydisperse suspensions. These mixtures consist of small solid particles that

* Corresponding author. Tel.: +56 41 203126; fax: +56 41 522055.
E-mail addresses: rburger@ing-mat.udec.cl (R. Bürger), donat@uv.es (R. Donat), mulet@uv.es (P. Mulet), cvega@uninorte.edu.co (C.A. Vega).

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belong to a number N of species that may differ in size or density, and which are dispersed in a viscous fluid. We will herein only consider particles of the same density. If ϕ_i denotes the volume fraction of particle species i having diameter D_i, where we assume that $D_1 > D_2 > \cdots > D_N$, and v_i is the phase velocity of species i, then the continuity equations of the N species are $\partial_t\phi_i + \partial_x(\phi_i v_i) = 0$, $i = 1,\ldots,N$, where t is time and x is depth. The velocities $v_1,\ldots,v_N$ are assumed to be given functions of the vector $\Phi := \Phi(x,t) := (\phi_1(x,t),\ldots,\phi_N(x,t))^{\mathrm{T}}$ of local concentrations. This yields nonlinear, strongly coupled systems of conservation laws of the type

$$\partial_t\Phi + \partial_x\mathbf{f}(\Phi) = 0, \quad \mathbf{f}(\Phi) := (f_1(\Phi),\ldots,f_N(\Phi))^{\mathrm{T}}, \quad f_i(\Phi) := \phi_i v_i(\Phi), \quad i = 1,\ldots,N. \tag{1.1}$$

We seek solutions $\Phi = \Phi(x,t)$ that take values in $\Phi \in \overline{\mathcal{D}}_{\phi_{\max}} \subset \mathbb{R}^N$, where $\overline{\mathcal{D}}_{\phi_{\max}}$ is the closure of the set

$$\mathcal{D}_{\phi_{\max}} := \left\{\Phi \in \mathbb{R}^N : \phi_1 > 0,\ldots,\phi_N > 0, \phi := \phi_1 + \ldots + \phi_N < \phi_{\max}\right\}.$$

The parameter $0 < \phi_{\max} \leqslant 1$ stands for a given maximum solids concentration. For batch settling of a suspension in a column of height L, (1.1) is defined on $\Omega_T := \{(x,t) \in \mathbb{R}^2 | 0 \leqslant x \leqslant L, 0 \leqslant t \leqslant T\}$ for a given final time $T > 0$ along with the initial condition

$$\Phi(x,0) = \Phi^0(x) = \left(\phi_1^0(x),\ldots,\phi_N^0(x)\right)^{\mathrm{T}}, \quad \Phi^0(x) \in \overline{\mathcal{D}}_{\phi_{\max}}, \quad x \in [0,L] \tag{1.2}$$

and the zero-flux boundary conditions

$$\mathbf{f}|_{x=0} = \mathbf{f}|_{x=L} = 0. \tag{1.3}$$

Several choices of v_i ("models"), or equivalently, of the fluxes f_i, as functions of Φ, and depending on the vector of normalized particle sizes $\mathbf{d} := (d_1,\ldots,d_N)^{\mathrm{T}}$, where $d_i := D_i/D_1$ for $i = 1,\ldots,N$, have been proposed in the literature. We restrict ourselves to the two models due to Masliyah [35] and Lockett and Bassoon [34] ("MLB model") and Höfler and Schwarzer [10,29,30] ("HS model"), respectively. It was recently shown in [9] that both models are strictly hyperbolic for all $\Phi \in \mathcal{D}_{\phi_{\max}}$, for arbitrary N, and under easily verifiable, mild restrictions on certain model-specific parameters and the smallest normalized particle size d_N. The key structural property of both models, which led to these results, consists in that the fluxes f_i do not depend on each of the N components of Φ in an individual way, but only on a small number $m \ll N$ ($m = 2$ and $m = 3$ for the MLB and HS models, respectively) of scalar functions of Φ. Therefore, the Jacobian $\mathcal{J}_{\mathbf{f}}(\Phi)$ of the flux vector of (1.1) is a rank-m perturbation of a diagonal matrix. The analysis of [9] also provides sharp bounds of the eigenvalues of $\mathcal{J}_{\mathbf{f}}(\Phi)$. This information eventually permits us to numerically calculate the eigenvalues and corresponding eigenvectors of $\mathcal{J}_{\mathbf{f}}(\Phi)$ with acceptable effort. Numerical simulations with low-order schemes were presented in [9], but it was conjectured that this characteristic (or spectral) information could be employed advantageously for the implementation of high-resolution schemes.

It is the purpose of this work to demonstrate that very efficient high-order accurate weighted essentially non-oscillatory (WENO) schemes for the numerical solution of (1.1)–(1.3) can indeed be constructed by incorporating characteristic information related to (1.1). This information is available due to the recent hyperbolicity analysis made in [9], and can be incorporated in various ways. Specifically, we use the results in [9] in order to provide a good estimation of the viscosity coefficient in a Lax–Friedrichs-type flux splitting. This allows to construct high resolution component-wise WENO schemes, akin to those proposed in [49] for the Multiclass Lighthill–Whitham–Richards (MCLWR) models in traffic flow. In addition, the full spectral decomposition of $\mathcal{J}_{\mathbf{f}}(\Phi)$, which can be numerically computed at each cell interface thanks to the analysis in [9], can be used in order to obtain *characteristic-based* WENO schemes, for which the WENO reconstruction procedure is applied to the local characteristic variables and fluxes at each cell-interface. When combined with a strong stability preserving (SSP) Runge–Kutta-type time discretization, the resulting SSP-WENO-SPEC schemes are shown to be extremely robust in a number of numerical experiments concerning the MLB and HS models, including several properties specific to the present application such as non-negativity of the solution, almost avoidance of overshoots of the numerical total density ϕ beyond $\phi_{\max}$, and accurate rendering of stationary kinematic shocks that separate sediment layers of different composition.

1.2. Related work

WENO-type spatial flux reconstructions, which emerged from earlier essentially non-oscillatory (ENO) schemes, have become a well-established, versatile tool for the construction of high-resolution conservative schemes in numerous applications. The first WENO scheme, of third-order accuracy, was introduced by Liu, Osher and Chan in [33], while a general framework to construct WENO schemes of arbitrary order of accuracy was provided by Jiang and Shu [31]. We refer to Shu [44,45] for further details, applications, and references. If applied to a system of conservation laws, the WENO procedure will produce a spatially semi-discrete system of ODE, for which a discretization in time can be chosen separately [43]. A suitable choice are total variation diminishing Runge–Kutta schemes [21,44], also known as strong stability preserving (SSP) methods [22], because of their favorable stability properties.

While WENO-based high-resolution shock-capturing schemes have been applied successfully to a wide range of convection-dominated problems [45], the polydisperse sedimentation models considered herein present some specific challenges for numerical simulation. These models belong to the wider class of multi-species kinematic flow models [14], which are characterized by a governing system of equations of the type (1.1) with explicit velocity functions $v_1,\ldots,v_N$ for a number