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A spectral FC solver for the compressible Navier–Stokes equations in general domains I: Explicit time-stepping

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ABSTRACT

We present a Fourier continuation (FC) algorithm for the solution of the fully nonlinear compressible Navier–Stokes equations in general spatial domains. The new scheme is based on the recently introduced accelerated FC method, which enables use of highly accurate Fourier expansions as the main building block of general-domain PDE solvers. Previous FC-based PDE solvers are restricted to linear scalar equations with constant coefficients. The FC methodology presented in this text thus constitutes a significant generalization of the previous FC schemes, as it yields general-domain FC solvers for nonlinear systems of PDEs. While not restricted to periodic boundary conditions and therefore applicable to general boundary value problems on arbitrary domains, the proposed algorithm inherits many of the highly desirable properties arising from rapidly convergent Fourier expansions, including high-order convergence, essentially spectrally accurate dispersion relations, and much milder CFL constraints than those imposed by polynomial-based spectral methods—since, for example, the spectral radius of the FC first derivative grows linearly with the number of spatial discretization points. We demonstrate the accuracy and optimal parallel efficiency of the algorithm in a variety of scientific and engineering contexts relevant to fluid-dynamics and nonlinear acoustics.

1. Introduction

The Fourier continuation (FC) methodology [12,31] was recently introduced as a central element of a class of unconditionally stable and dispersionless alternating-direction implicit PDE solvers, the FC-AD methods, for *linear constant-coefficient* hyperbolic, parabolic and elliptic PDEs in general d-dimensional domains ($d = 2,3,\ldots$). At the heart of that methodology lies an accelerated periodic-continuation algorithm—the FC(Gram) scheme—which enables Fourier approximation of non-periodic functions without the highly detrimental slow convergence and Gibbs ringing inherent in standard Fourier expansions of non-periodic functions. As utilized in [12,31], the FC(Gram) method allows for fast, highly accurate spectral solution of the types of ODEs that arise in the application of alternating direction constant-coefficient PDE solvers—with high-order accuracy up to and including domain boundaries; a generalization of this methodology to linear PDEs with variable coefficients is presented in [13]. In this paper, in turn, we present a new FC-based *explicit* solver for the fully-nonlinear compressible Navier–Stokes equations. As discussed in what follows and demonstrated in the body of this text, this solver, which relies on use of Cartesian meshes (and, thus, is only subject to mild CFL constraints), which delivers spectral accuracy in domain interiors and fifth-order accuracy at domain boundaries, and which is amenable to highly effective, perfectly-scaling parallelization, can significantly outperform previous solvers for the problems under consideration—by orders of magnitude in accuracy and computing times.

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The value of spectrally accurate numerical solvers for time-dependent PDEs is well established. Spectral methods can handle large-scale configurations by means of relatively coarse discretizations, and thus they help keep computational costs under control. And, for long-time simulations in which complicated flow structures or acoustic waves must propagate or be convected for long distances, they possess the crucial ability to faithfully preserve the dispersion characteristics of the corresponding continuous problems. Unfortunately, however, previous spectral solvers are subject to a number of significant constraints, including time-step restrictions for stability in polynomial spectral methods (which arise from corresponding CFL conditions), geometric and periodicity constraints for the pure Fourier-based methods (to avoid the occurrence of the Gibbs phenomenon), and, for PDEs defined on general domains, the difficulties arising from the requisite uses of domain mappings into rectangular domains, which are, in general, prohibitively complex. The FC solver introduced in this paper, in contrast, which is based on use of equispaced Cartesian meshes as well as rapidly-computable, one-dimensional FC-based Fourier expansions for non-periodic functions, satisfactorily bypasses all of these restrictions, while retaining the highly desirable characteristics associated with classical spectral methods.

1.1. CFL constraints: Finite-Element/Finite-Volume solvers

The use of regularly spaced grids gives rise to favorable CFL constraints—since the numerical differentiation operators associated with such grids generally possess spectral radii which grow only linearly with the number of grid points in each coordinate direction. In contrast, the spectral radii of differentiation operators based on non-uniformly spaced structured grids (polynomial-based spectral methods, for example) generally grow super-linearly—owing to point clustering near boundaries—and therefore give rise to stringent CFL conditions. High-order methods based on unstructured meshes, similarly, often give rise to differentiation operators with spectral radii which grow not only with the mesh fineness, but also with the approximation order. The advantages arising from avoidance of such challenging CFL constraints can be very significant. For example, in a case study considered in Section 5.1, which was previously put forth in reference [17], the FC solver attains the given accuracy in less than 1/200 times the computing time required by the fastest hybrid Finite-Volume/Discontinuous-Galerkin high-order solver presented in that reference.

1.2. Non-periodicity, FFTs, parallel scaling

Although, for periodic problems, the Fourier collocation method offers unchallenged accuracy and perfect dispersion, this method is not applicable, without tremendous degradation, to problems in which the solution is not a periodic function in a rectangular domain. Moreover, although the Fast Fourier Transform (FFT) algorithm provides the Fourier collocation method with a computational complexity of $O(N\log N)$, where N is the size of discretization mesh, the $\log N$ factor can be very significant for large-scale problems. While there have been a number of previous attempts to eliminate the difficulties arising from the Gibbs phenomenon and thus render the Fourier collocation method applicable to boundary value problems (see, for example, [9,14,21,23] and the references therein), it appears that such methods have not lead, thus far, to efficient numerical solvers for time-dependent PDEs in general domains. The FC solver presented in this paper, in contrast, allows for use of Fourier-collocation-based derivative approximations in complex geometries with high-order accuracy and essentially no numerical dispersion; it can effectively offer linear computational complexity (without an additional logarithmic factor, see Remark 2.6); and, unlike the FFT algorithm itself, it exhibits optimal parallel scaling (Section 6).

1.3. Finite-difference solvers, numerical dispersion

The majority of high-order PDE solvers are based on use of finite differences. The development of high-order finite difference solvers is by no means a trivial task, however: the absence of solution values outside the computational domain requires that the finite difference stencils must be made increasingly one-sided as regions near domain boundaries are approached. In general, the simple procedure based on use of high-order centered difference methods in the interior of the domain and equally high-order biased stencils near the boundary does not produce stable solvers. This problem has been addressed through the development of Padé (or compact) schemes (see, e.g., [5,30])—which, at the additional computational expense of inverting banded matrices along the coordinate directions, reduce stencil sizes, and thus decrease the numbers of near-boundary points that must be treated. Another approach to tackle this problem is provided by the finite-difference methods associated with Summation By Parts (SBP) operators (see, e.g., [34,36,39,40]), which make use of energy estimates for semi-discrete problems to obtain near-boundary derivative operators that result in stable PDE solvers. Both the Padé and SBP methods must sacrifice some accuracy near the boundary to gain stability. In [5,30], for example, we find Padé schemes that have interior accuracies of sixth and fourth order respectively and both reduce to third-order on the boundary. The contribution [38] discourages use of SBP methods whose boundary order is more than half the interior order: the SBP algorithms found in recent literature have interior/boundary orders of 6/3 and 8/4. The results in Section 3.3 and, in particular Fig. 5, show that the FC method introduced in this paper, which is accurate to order five at boundaries and essentially spectral at domain interiors, provides significantly better dispersion curves (and commensurately better accuracies) than the high-order SBP and compact schemes.