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Adaptive variational multiscale methods for incompressible flow based on two local Gauss integrations ☆

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ABSTRACT

We consider variational multiscale (VMS) methods with *h*-adaptive technique for the stationary incompressible Navier–Stokes equations. The natural combination of VMS with adaptive strategy retains the best features of both methods and overcomes many of their deficits. A reliable a posteriori projection error estimator is derived, which can be computed by two local Gauss integrations at the element level. Finally, some numerical tests are presented to illustrate the method's efficiency.

1. Introduction

In the numerical simulation of incompressible flows, there are still many challenges, such as, how to control the accuracy of a numerical approximation for the solutions, which may be degraded by the local singularities or the singularity in the computational domain. Since the work by Babuska and Rheinboldt [1,2], adaptive control based on a posteriori error estimates has become very attractive. Many researchers pay their attention on the field of a posteriori error estimators and have got lots of good results in the last few decades, for example, [3–5] derive the residual-based a posteriori error estimate. Deriving a posteriori error estimates for the Stokes equations also has received much attention (see [4,6–8] and so on), for the Navier–Stokes equation, see [9]. Many people also develop some other methods, like, the estimators based on the element residual, based on evaluating integrals of the residuals or associated with spatial averages. Besides, the recovery type error estimators are discussed in [10–16], recently.

Variational multiscale methods are designed to deal with incompressible flow, which define the large scales in a different way, namely by projection into appropriate subspaces, see Guermond [17], Hughes et al. [18–20] and Layton [21], and other literatures on VMS methods [21–26]. The idea of two local Gauss integrations has been considered to deal with the variational multiscale methods (such as [27]).

There are also some researchers trying to combine the adaptive strategy with stabilization method, such as [9,28]. In this paper, we try to combine VMS with *h*-adaptive technique, and the combination is particularly efficient and combines the best algorithmic features of each. Although, a posteriori error estimator is derived based on a projection operator, but by using two local Gauss integrations, this estimator can be computed easily at the element level. The global upper bound for the error of the finite element discretization is yielded follows some assumptions.

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The outline of the paper is as follows. Section 2 introduces the governing equations, the notations and some well-known results used for variational multiscale methods of the Navier–Stokes problem throughout the paper. The posteriori error estimator based on local projection is presented in Section 3, and the equivalent version based on two local Gauss integrations is derived. In Section 4, some numerical simulations are presented to illustrate the efficiency of the combination of VMS with adaptive strategy. We finish with a short conclusion in Section 5.

2. Governing equations

We consider the incompressible flows

$$\begin{aligned} &-\nu\Delta u + (u\cdot\nabla)u + \nabla p = f \quad \text{in} \quad \Omega,\\ &\nabla\cdot u = 0 \quad \text{in} \quad \Omega,\\ &u = 0 \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{2.1}$$

where Ω represents a polyhedral domain in R^d, $d = 2,3$ with boundary $\partial\Omega$, u the velocity vector, p the pressure, f the prescribed body force, and $\nu > 0$ the kinematic viscosity, which is inversely proportional to the Reynolds number Re.

The standard variational formulation of (2.1) is given by: find $(u,p) \in (V,S)$ satisfying

$$\mathbf{B}(u,p;v,q) + b(u,u,v) = (f,v) \quad \forall (v,q) \in (V,S), \tag{2.2}$$

where

$$\begin{aligned} &V = H_0^1(\Omega)^d \quad \text{and} \quad S = L_0^2(\Omega) = \left\{ q \in L^2(\Omega); \int_\Omega q\, d\mathbf{x} = 0 \right\},\\ &\mathbf{B}(u,p;v,q) = \nu(\nabla u, \nabla v) - (\nabla\cdot v, p) + (\nabla\cdot u, q), \quad b(u,u,v) = ((u\cdot\nabla u), v), \end{aligned}$$

with $(\cdot,\cdot)$ the inner product in $L^2(\Omega)$ or in its vector value versions. The norm and seminorm in $H^k(\Omega)^d$ are denoted by $\|\cdot\|_k$ and $|\cdot|_k$, respectively. $H_0^1(\Omega)$ will denote the closure of C_0^∞ with respect to the norm $\|\cdot\|_1$. The space V is equipped with the norm $\|\nabla\cdot\|_0$ or its equivalent norm $\|\cdot\|_1$ due to the Poincaré inequality.

For the finite element discretization, let τ_h be the regular triangulations of the domain Ω, and define the mesh parameter $h = \max_{T\in\tau_h}\{\text{diam}(T)\}$. We choose the conforming velocity–pressure finite element space $(V_h,S_h) \subset (V,S)$ satisfying the discrete inf-sup condition

$$\inf_{q_h\in S_h} \sup_{v_h\in V_h} \frac{(q_h, \nabla\cdot v_h)}{\|q_h\|_0 \|\nabla v_h\|_0} \geqslant \beta > 0, \tag{2.3}$$

where β is independent of h. Here we consider the Taylor–Hood elements (see [29,30]):

$$\begin{aligned} &V_h = \left\{ u_h \in C(\Omega)^d | u_h|_T \in P_2(T)^d, \quad \forall T \in \tau_h \right\},\\ &S_h = \left\{ q_h \in C(\Omega) | q_h|_T \in P_1(T), \quad \forall T \in \tau_h \right\}, \end{aligned}$$

where $P_k(T)$, $k = 1,2$ is the space of kth-order polynomials on T. We will also need the piecewise constant space

$$R_0 = \{ v_h \in L^2(\Omega) | v_h|_T \in P_0(T), \quad \forall T \in \tau_h \},$$

where $P_0(T)$ is the space of all constant polynomials on T.

Throughout this paper, we shall use the letter C (with or without subscripts) to denote a generic positive constant which may stand for different values at its different occurrences but that remains independent of the mesh parameter h.

Then, Galerkin finite element discretization of (2.2) is given by: find $(u_h,p_h) \in (V_h,S_h)$ satisfying

$$\mathbf{B}(u_h,p_h;v_h,q_h) + b(u_h,u_h,v_h) = (f,v_h) \quad \forall (v_h,q_h) \in (V_h,S_h). \tag{2.4}$$

Because of inequality (2.3), problem (2.4) has a unique solution and the error estimate

$$\|\nabla(u-u_h)\|_0 + \|p-p_h\|_0 \leqslant Ch^2\{\|u\|_3 + \|p\|_2\}, \tag{2.5}$$

holds provided $(u,p) \in (H^3(\Omega)^d, H^2(\Omega))$.

As we know, the Galerkin finite element discretization (2.4) is unstable in the case of higher Reynolds number (or smaller viscosity). Therefore, stabilization becomes necessary. We firstly consider a common version of VMS methods which was proposed in [21] for the steady case. We define two spaces $L = L^2(\Omega)^{d\times d}$ and $L_h = R_0(\Omega)^{d\times d}$, the latter is defined on the same grid as X_h for the velocity deformation tensor. The VMS we consider here is: find $(u_h,p_h) \in (V_h,S_h)$ and $g_h \in L_h$ satisfying

$$\begin{aligned} &(\nu+\alpha)a(u_h,v_h) - \alpha(g_h,\nabla v_h) + b(u_h,u_h,v_h) - d(p_h,v_h) = (f,v_h) \quad \forall v_h \in V_h,\\ &d(q_h,u_h) = 0 \quad \forall q_h \in S_h,\\ &(g_h - \nabla u_h, l_h) = 0 \quad \forall l_h \in L_h. \end{aligned} \tag{2.6}$$