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A family of MPFA finite-volume schemes with full pressure support for the general tensor pressure equation on cell-centered triangular grids

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ABSTRACT

A new family of cell-centered finite-volume schemes is presented for solving the general full-tensor pressure equation of subsurface flow in porous media on arbitary unstructured triangulations. The new schemes are flux continuous and have full pressure support (FPS) over each subcell with continuous pressure imposed across each control-volume sub-interface, in contrast to earlier formulations. The earlier methods are point-wise continuous in pressure and flux with triangle-pressure-support (TPS) which leads to a more limited quadrature range. An M-matrix analysis identifies bounding limits for the schemes to posses a local discrete maximum principle. Conditions for the schemes to be positive definite are also derived.

A range of computational examples are presented for unstructured triangular grids, including highly irregular grids, and the new FPS schemes are compared against the earlier pointwise continuous TPS formulations. The earlier pointwise TPS methods can induce strong spurious oscillations for problems involving strong full-tensor anisotropy where the M-matrix conditions are violated, and can lead to decoupled solutions in such cases. Unstructured cell-centered decoupling is investigated. In contrast to TPS, the new FPS formulation leads to well resolved solutions that are essentially free of spurious oscillations.

A substantial degree of improved convergence behavior, for both pressure and velocity, is also observed in all convergence tests. This is particularly important for problems involving high anisotropy ratios. Also the new formulation proves to be highly beneficial for an upscaling example, where enhancement of convergence is highly significant for certain quadrature points, clearly demonstrating further advantages of the new formulation.

1. Introduction

Rapid variation in permeability with strong anisotropy are common features in subsurface reservoirs. Numerical discretization methods must be able to accurately model fluid flow by the use of flexible grids, in order to resolve various important complex geological features in the reservoir. In recent years several discretization methods have been developed that can treat unstructured grids in combination with discontinuous and anisotropic permeability fields [1–10]. In particular the flux-continuous finite volume schemes of [4] are termed control-volume distributed (CVD) and include a family of methods,

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while the method of [11] is termed a multi point flux approximation (MPFA) method. These schemes are related to the mixed finite element (MFE) method, e.g., [12–17].

The MFE and related methods solve for both control-volume pressure and cell face velocities leading to a globally coupled indefinite linear system (saddle point problem), while the more efficient CVD(MPFA) methods only solve for control-volume pressure and involve a locally coupled algebraic system for the fluxes that yield a consistent continuous approximation, while only requiring one third the number of degrees of freedom of the mixed method when compared on a structured 2-D grid (and a quarter in 3-D). The latter methods are clearly advantageous, particularly for time-dependent problems, as the extra degrees of freedom required by the mixed method add further computational complexity and a severe penalty to simulation costs. The relationship between CVD(MPFA) and MFE was first presented in [4] for flux quadrature point $q = 1$ and used in a convergence proof in [18]. The relationship between families of schemes in physical space (which are not generally symmetric), and transform space with normalised quadrature range $0 < q \leqslant 1$, is presented in [19] for quadrilateral grids, where the relationship between both formulations and MFE is also presented for general q. An MFE method closely related to CVD(MPFA) is presented in [20]. Similarities between MPFA methods and certain (control volume) MFE methods are investigated in [21]. Coupling of the methods with higher-order convective flow approximation is presented in [22,23] and with multi-dimensional upwind schemes in [24].

When formulating a finite-volume pressure equation scheme, continuous pressure and normal flux are key physical constraints that must be imposed at control-volume interfaces, across which strong discontinuities in permeability can occur. Earlier flux-continuous CVD(MPFA) schemes have triangle-pressure-support (TPS) e.g. [25,4,5,19,10] and [2,11,6,1] which are only point-wise continuous in pressure and flux. Numerical tests of MPFA methods [25,26,5,10] indicate that the convergence rates for pressure and flux are of the same order as for the MFE methods when grid refinement is uniform. The work of [5] has provided the first study of *convergence of the family* of schemes and established improved convergence for certain values of the quadrature parametrization q for lower anisotropy ratios. Positive definiteness and M-matrix conditions are given in [25,27,4,19] for families of CVD(MPFA) schemes. The generalised base quadrilateral grid scheme ($q = 1$), is shown to be symmetric positive definite SPD for a full tensor when a subcell transform space formulation is employed [19,27,4]. A Symmetric Positive Definite family of schemes is presented on cell centered triangular grids in [10] which includes the special case of a unique SPD physical space scheme defined when flux quadrature points are placed at one third of each triangle edge measured from each vertex. For full-tensor cases involving large anisotropy ratios these pointwise continuous TPS methods violate M-matrix conditions and are shown to yield decoupled solutions with strong spurious oscillations in the resulting numerical pressure fields.

A new family of control-volume distributed MPFA schemes for quadrilateral grids with full-pressure-support (FPS), such that full pressure continuity is imposed across control-volume sub-faces is introduced by Edwards and Zheng [28] and extended to cell-vertex unstructured grids in [29]. A double-family cell-vertex FPS formulation is presented for quadrilateral and triangle grids in [30] together with M-matrix conditions for both grid-types, and optimal schemes are identified via anisotropic quadrature rules and by anisotropic triangulation. The FPS schemes exhibit a considerable improvement in stability and yield well resolved solutions that are essentially free of spurious oscillations. Another recent MPFA scheme, based on a somewhat different starting point, but still enjoying full pressure continuity, was presented by Le Potier in [7]. Further schemes designed to compute solutions free from spurious oscillations are presented in [31–33,7,34].

In this paper a new family of locally conservative, flux-continuous, finite-volume schemes is presented for solving the general tensor pressure equation on unstructured cell-centered triangular grids. These new sub-cell schemes have full pressure support (FPS), in contrast to the earlier families of flux-continuous TPS schemes on cell-centered triangular grids.

An M-matrix analysis is presented for the cell-centered unstructured formulation that identifies bounding limits on tensor off-diagonal coefficients in order for the schemes to posses a local discrete maximum principle. Conditions for the schemes to be positive definite are also derived with respect to the symmetric part of the local tensor.

The new schemes are tested on a number of cases, including highly irregular triangular grids, and compared with the non-symmetric MPFA scheme of [2] and the more recent SPD MPFA TPS schemes introduced in [10]. A substantial degree of improved convergence behavior is achieved by the new FPS schemes, for both pressure and velocity, in all cases considered. This is particularly important for problems involving high anisotropy ratios, where the cell-centered triangular grid TPS methods typically violate M-matrix conditions and induce strong spurious oscillations in the solution. The instability is shown to be consistent with decoupled solution modes in such cases. In contrast the new methods compute well resolved pressure fields with relatively few spurious oscillations. The new formulation also proves to be highly beneficial for an upscaling example, where enhancement of convergence is highly significant for certain quadrature points, clearly demonstrating the usefulness of the new formulation.

The paper is organised as follows. Section 2 gives a description of the single phase flow problem encountered in reservoir simulation with respect to the general tensor pressure equation. Section 3 presents necessary notation as well as the underlying principles of MPFA flux approximations. Details of the new cell-centered flux-continuous triangular grid schemes with full pressure support are presented in Section 4 via the specific example of a three triangle cluster. M-matrix conditions are derived in Section 5. The occurance of decoupled solution modes resulting from the earlier TPS formulation are considered in Section 6, where it is also observed that a monotone matrix is sufficient to prevent such modes. Conditions for the formulation to be positive definite are derived in Section 7. Numerical examples are presented in Section 8, that illustrate benefits of the new FPS scheme. Finally, conclusions follow in Section 9.