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Simplex free adaptive tree fast sweeping and evolution methods for solving level set equations in arbitrary dimension

Thomas C. Cecil [a,*], Stanley J. Osher [b], Jianliang Qian [b,1]

[a] ICES, University of Texas, Austin, 1 University Station, C0200, Austin, TX 78712, United States
[b] Department of Mathematics, University of California, Los Angeles, 405 Hilgard Avenue, Los Angeles, CA 90095-1555, United States

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Abstract

We introduce simplex free adaptive tree numerical methods for solving static and time-dependent Hamilton–Jacobi equations arising in level set problems in arbitrary dimension. The data structure upon which our method is built in a generalized n-dimensional binary tree, but it does not require the complicated splitting of cubes into simplices (aka generalized n-dimensional triangles or hypertetrahedrons) that current tree-based methods require. It has enough simplicity that minor variants of standard numerical Hamiltonians developed for uniform grids can be applied, yielding consistent, monotone, convergent schemes. Combined with the fast sweeping strategy, the resulting tree-based methods are highly efficient and accurate. Thus, without changing more than a few lines of code when changing dimension, we have obtained results for calculations in up to $n = 7$ dimensions.

Keywords: Numerical methods; Binary trees; Adaptive methods; Level sets; Hamilton–Jacobi; Fast sweeping

1. Introduction

In this paper, we present a simplex free adaptive tree numerical method for solving static and time-dependent Hamilton–Jacobi partial differential equations (H–J PDEs) arising in level set problems in arbitrary dimension. The method's adaptivity increases resolution near the interface being studied, and simplifies previous successful tree-based implementations, allowing for its extension to arbitrary dimension without an increase in the complexity of function reconstruction, which is a necessary part of finding spatial derivatives needed in solving the PDEs.

Applications in higher dimensions requiring adaptive meshes to resolve fine details arise in numerous fields. In [23,14,8,30] multi-valued solutions to H–J equations were found by replacing a single-valued solution with

* Corresponding author. Tel.: +1 512 2327761.
E-mail addresses: tcecil@ices.utexas.edu (T.C. Cecil), sjo@math.ucla.edu (S.J. Osher), qian@math.ucla.edu, qian@math.wichita.edu (J. Qian).
[1] Present address: Department of Mathematics and Statistics, Wichita State University, Wichita, KS 67260-0033, United States.

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the level set (or intersection of level sets) of a higher-dimensional function. This idea was also used in [5,7] to study interfaces with codimension > 1. In [28], the incompressible Euler equations were studied. In [33,34], a level set formulation was used to solve problems arising in mathematical finance, where high-dimensional issues are routinely encountered. Even for codimension-1 problems in 3D, there is still a desire to find implementable adaptive methods to resolve fine details, such as in the segmentation of the human brain, or other applications involving highly curved surfaces such as Wulff crystals. See [24,32] for a wide range of physical problems to which level set methods are applied.

Since the introduction of level set methods for interface tracking [22], there has been work done in an attempt to reduce the component of the computational portion of the method subject to the most criticism: the necessity of extra dimensions. Within a few years following [22] narrow band methods were proposed that reduced the computational complexity by resolving the level set function only near the interface being tracked [1,39,26]. These methods were able to use the well established, convergent, finite difference schemes available to uniform grids.

However, these narrow band methods did not reduce the storage requirements, limiting them to the same resolutions which uniform grids were restricted. Following this, tree-based methods were introduced, allowing for adaptivity of the mesh near the interface, while not sacrificing too much complexity [35,20,10,18]. The tree data structure used in these methods was well understood by the computer science community, and thus data storage and retrieval were able to be carried out in an efficient manner. However, the non-uniformity of the mesh required new schemes to be developed for the PDEs to be solved. In some cases semi-Lagrangian CIR [9] schemes were used for time-dependent level set equations. These schemes have some drawbacks, though. Firstly, they are only provably convergent for hyperbolic problems, and many level set PDEs involve mean curvature or are otherwise parabolic in nature. Secondly, they require a backtracking along characteristics and an interpolation at an arbitrary point within the domain. This interpolation is a delicate process that requires the division of the domain into simplices, which can become complicated in higher dimensions [20]. In [18], CIR was used for advection of values stored at cell corners, and cell centered data was stored for the pressure equation in Navier–Stokes, where a one-point (constant within each cell) interpolation technique was used to avoid apparent complexities, and to preserve the symmetry of the discretization. In addition, for the eikonal equation used to maintain the signed distance property of the level set function, [18] used some special treatment at T-junctions in the context of the fast marching spirit [38].

There have been other local level set methods [19,36,4,13,3] proposed which range from variants of AMR to using tubes of uniformly spaced grid points near the interface. Some of the methods approach the complexity of [26], eliminating the need to store the unused grid points away from the interface of interest. They also allow for the standard finite difference schemes to be used as the grid is uniform near the interface. However, with these gains comes additional complexity in implementation, and it should be noted that the successive improvements and acceptance of the tree-based methods in various applications are testaments to their facility and usefulness.

In this paper, we introduce a tree-based method that retains the advantages of the previous tree-based algorithms, such as having a well studied and understood data structure, while avoiding the drawbacks of having inconsistent schemes requiring n-dimensional simplices and interpolation. Thus we are able to use the standard numerical Hamiltonians derived for uniform grids (modified slightly) which result in consistent, monotone, convergent numerical methods. Combined with the fast sweeping strategy [41,37,15,29], the resulting tree-based methods are highly efficient and accurate.

The paper consists of a brief overview of the tree data structure, followed by a discussion of the numerical schemes for static H–J equations, and then time-dependent H–J equations. Finally, numerical results are given for codimension-1, codimension-2, and codimension-n problems.

2. Tree data structure

In this section, we describe the tree data structure used. We use a generalized binary tree (e.g., quadtree in 2D, octree in 3D, etc.) data structure, details of which can be found in numerous computer science texts [17,31,12]. We describe the portions of the implementation that are specific to our problem of solving a PDE in a bounded spatial domain.