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A convergence study for SPDEs using combined Polynomial Chaos and Dynamically-Orthogonal schemes

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ABSTRACT

We study the convergence properties of the recently developed Dynamically Orthogonal (DO) field equations [1] in comparison with the Polynomial Chaos (PC) method. To this end, we consider a series of one-dimensional prototype SPDEs, whose solution can be expressed analytically, and which are associated with both linear (advection equation) and nonlinear (Burgers equation) problems with excitations that lead to unimodal and strongly bi-modal distributions. We also propose a hybrid approach to tackle the singular limit of the DO equations for the case of deterministic initial conditions. The results reveal that the DO method converges exponentially fast with respect to the number of modes (for the problems considered) giving same levels of computational accuracy comparable with the PC method but (in many cases) with substantially smaller computational cost compared to stochastic collocation, especially when the involved parametric space is high-dimensional.

1. Introduction

Recently, there has been a growing interest in quantifying parametric uncertainty in mathematical physics problems through the probabilistic framework. Such problems are often described by stochastic partial differential equations (SPDEs), and they arise in fluid mechanics, solid mechanics, wave propagation through random media [2–4], random vibrations [5–7], etc. The source of stochasticity in all the above cases includes uncertainty in physical parameters, initial and/or boundary conditions, random excitations, etc. All these stochastic elements may be modeled as random processes or random variables. Several methods have been developed to study SPDEs, including the Monte Carlo (MC) method and its variants and, more recently, Polynomial Chaos (PC) and its variants. One of the often neglected issue in developing new numerical methods is the study of the convergence properties of the method, which is especially important for stochastic solutions that lack regularity and they are typically high-dimensional.

The Polynomial Chaos (PC) method was developed in [8] in the context of the Wiener–Hermite polynomial chaos expansion. The stochastic processes are represented by a series of Hermite polynomials in terms of random variables. A Galerkin projection of the governing equations to the low-dimensional subspace spanned by Hermite polynomials yields a set of deterministic equations. PC has been applied to many problems including structural mechanics [9–11], fluid mechanics [12–16], etc. The generalized polynomial chaos developed by [14,17] employ non-Hermite polynomials to improve efficiency for a wider class of nonlinear problems. A computationally efficient version of PC is the probabilistic collocation method (PCM; also referred to as stochastic collocation), which exhibits fast convergence rates with increasing order of the

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expansions, provided that solutions are sufficiently smooth in the parametric space [18–20]. In particular, the multi-element PCM (ME–PCM) is very effective for problems with parametric discontinuities [20].

A new approach, called Dynamically Orthogonal (DO) method, was developed in [1]; the idea is to represent the solution in a more general expansion, i.e.,

$$u(x,t;\omega) = \bar{u}(x,t) + \sum_{i=1}^{N} Y_i(t;\omega)u_i(x,t),$$

where $Y_i(t;\omega)$ are stochastic processes, $u_i(x,t)$ orthonormal fields and $\bar{u}(x,t)$ is the mean. The time dependence on *both* the stochastic coefficients and the basis fields makes the above representation very flexible for the representation of strongly transient, non-stationary responses. However, this same property makes the representation redundant and the derivation of well-posed equations for all the quantities involved is not a straightforward problem. In [1] it was illustrated how this redundancy can be overcome by adopting a natural constraint: *the dynamical orthogonality condition*. It was shown that using this condition a set of evolution equations for the $Y_i(t;\omega), u_i(x,t)$ and $\bar{u}(x,t)$ can be derived. These derived field equations are consistent with existing methods such as proper orthogonal decomposition method (POD) and PC.

From a computational point of view the evolution of uncertainty using the DO framework is performed by solving a set of $(N+1)$ deterministic PDEs together with N (ordinary) stochastic differential equations (SDE). The system of PDEs describes the evolution of the mean field and the basis elements that define the stochastic subspace where uncertainty 'lives'. The SDE on the other hand defines how the stochasticity will be evolved within the reduced-order stochastic subspace. In the limit of very small uncertainty the DO equations reach a singular limit where the modes evolve independently from the statistics within the subspace. This limit may create important numerical problems since it involves the calculation of ratios of very small moment quantities. From a practical point of view the above situation can be very important especially in problems involving deterministic initial conditions. In such a case (deterministic initial state) another issue rises and this is the initialization of the stochastic subspace (since any initial choice of modes is allowable).

To tackle this issue we develop a hybrid method to overcome this singularity by combining the PC method with the DO method. Initially, the SPDEs are solved by the PC (e.g. via PCM or ME–PCM), and as the stochasticity develops we switch over to the DO method. This hybrid approach also gives us a set of modes, which are used to initiate the stochastic subspace. We consider three one-dimensional SPDEs which are associated with linear (advection equation) and nonlinear (Burgers and diffusion equation). For the advection equation the solution can be expressed analytically. For Burgers equation we consider two other cases: one has the analytical solution with excitation functions that lead to unimodal and bi-modal distributions while the other has random forcing. For the diffusion equation we consider the unsteady heat equation with uncertain inputs in heat conductivity that is multi-dimensional in parametric space. In particular, we examine the convergence of the new hybrid method and compare its accuracy and the efficiency with the PCM, which is the golden standard in uncertainty quantification at the present time.

The remaining part of the paper is organized as follows. In Section 2 we briefly review the DO representation and the corresponding evolution equations. Subsequently, we present the hybrid method of PC and DO. In the following sections, the hybrid method is applied to two SPDEs. In Section 3, a stochastic advection equation is considered; the exact formulas for the stochastic coefficients and deterministic basis are given and comparison with PCM is presented. In Section 4, two Burgers equations are considered. First, the Burgers solution is constructed given the stochastic coefficients and basis and the corresponding exact PDF is derived. Second, the Burgers equation is considered for testing the hybrid method and convergence with respect to the number of modes. In Section 5, the nonlinear diffusion equation is considered; it is multi-dimensional in parametric space with no exact solution and the convergence with respect to the number of modes is presented. We conclude the paper with a brief summary in Section 6.

2. An overview of the DO equations and a new hybrid DO-PC approach

We consider the following SPDE:

$$\frac{\partial u}{\partial t} = \mathscr{L}(u(t,x;\omega)), \quad x \in D, \quad \omega \in \Omega \tag{1a}$$

$$u(t_0,x;\omega) = u_0(x;\omega), \quad x \in D, \quad \omega \in \Omega \tag{1b}$$

$$\mathscr{B}[u(t,x;\omega)] = h(t,x;\omega), \quad x \in \partial D, \quad \omega \in \Omega, \tag{1c}$$

where $\mathscr{L}$ is a differential operator and $\mathscr{B}$ is a linear differential operator. D is a bounded domain in $\mathscr{R}^d$ where $d = 1, 2$, or 3.

2.1. Definitions

Let $(\Omega, \mathscr{F}, P)$ be a probability space, where Ω is the sample space, $\mathscr{F}$ is the σ-algebra of subsets of Ω, and P is a probability measure. For a random field $u(x,t;\omega)$, $\omega \in \Omega$, the expectation operator of u is defined as

$$\bar{u}(x,t) = E[u(x,t;\omega)] = \int_{\Omega} u(x,t;\omega)dP(\omega).$$