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A study of spectral element and discontinuous Galerkin methods for the Navier–Stokes equations in nonhydrostatic mesoscale atmospheric modeling: Equation sets and test cases ☆

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Abstract

We present spectral element (SE) and discontinuous Galerkin (DG) solutions of the Euler and compressible Navier–Stokes (NS) equations for stratified fluid flow which are of importance in nonhydrostatic mesoscale atmospheric modeling. We study three different forms of the governing equations using seven test cases. Three test cases involve flow over mountains which require the implementation of non-reflecting boundary conditions, while one test requires viscous terms (density current). Including viscous stresses into finite difference, finite element, or spectral element models poses no additional challenges; however, including these terms to either finite volume or discontinuous Galerkin models requires the introduction of additional machinery because these methods were originally designed for first-order operators. We use the local discontinuous Galerkin method to overcome this obstacle. The seven test cases show that all of our models yield good results. The main conclusion is that equation set 1 (non-conservation form) does not perform as well as sets 2 and 3 (conservation forms). For the density current (viscous), the SE and DG models using set 3 (mass and total energy) give less dissipative results than the other equation sets; based on these results we recommend set 3 for the development of future multiscale research codes. In addition, the fact that set 3 conserves both mass and energy up to machine precision motives us to pursue this equation set for the development of future mesoscale models. For the bubble and mountain tests, the DG models performed better. Based on these results and due to its conservation properties we recommend the DG method. In the worst case scenario, the DG models are 50% slower than the non-conservative SE models. In the best case scenario, the DG models are just as efficient as the conservative SE models.
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Keywords: Compressible flow; Euler; Lagrange; Legendre; Navier–Stokes; Nonhydrostatic; Viscous flow

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1. Introduction

The recent paradigm shift in high-performance computing (HPC) is forcing many numerical weather prediction (NWP) operational centers to rethink the numerical methods that their models are based on. For example, the current trend in distributed-memory computing has moved toward clusters based on hundreds of thousands of cheap, commodity-based processors; the top three fastest computers in the world in 2007 have 212,000 (Lawrence Livermore National Laboratory, USA), 65,000 (Forschungszentrum Juelich, Germany), and 14,000 (New Mexico Computing Applications Center, USA). It is expected that clusters comprised of millions of processors will appear very soon. In order to take full advantage of computers with such high processor counts requires exploring numerical methods that are local in nature, have a large on-processor operation count, and a small communication footprint. Local high-order methods like the spectral element and discontinuous Galerkin methods have all of these properties and for this reason they have been successfully applied to a variety of problems.

Spectral element (SE) methods combine the local domain decomposition property of finite element (FE) methods with the high-order accuracy and weak numerical dispersion of spectral methods. SE methods have shown promise in many areas of the geosciences including: seismic wave propagation [33], deep Earth flows [13], climate [53,14], ocean [28,38], and numerical weather prediction [21,22]. These methods are high-order FE methods where the interpolation and integration points are chosen to be the Legendre–Gauss–Lobatto points.

In contrast, discontinuous Galerkin (DG) methods combine the local domain decomposition property of FE methods, the high-order accuracy and weak numerical dispersion of spectral methods, and the conservation properties of finite volume (FV) methods; in essence, DG methods are the high-order generalization of FV methods. There are two distinct types of DG methods: nodal (see [20,23]) and modal (see [12,59]), but in the current study we only consider the nodal approach introduced in [20] which uses the same machinery developed for SE methods such as quadrilateral grids, tensor product basis functions, and Legendre–Gauss–Lobatto grid points. It has been only very recent (since 2000) that the DG method first appeared in geophysical fluid dynamics (GFD) applications. However, implementations of the DG method in GFD have remained primarily restricted to shallow water flow (see [44,35,2,20,10,12,37,40,34,23,24]). To date, there has been no published work on either SE or DG methods for nonhydrostatic mesoscale atmospheric applications.

However, doing something for the first time is not a sufficient reason for developing a new model – the new model must have attractive properties not offered by existing models. The high-order accuracy, geometric flexibility to use any grid, and the scalability of SE and DG methods on large processor count computers are sufficient reasons for exploring this new class of models.

Almost all nonhydrostatic mesoscale models currently in existence are based on the finite difference (FD) method; examples include the following list of models: [4,5,11,17,25,26,29–32,39,42,43,46,48,57,58], and [60]. Included in this list are models such as ARPS (University of Oklahoma), COAMPS (US Navy), LM (German Weather Service), MC2 (Environment Canada), MM5 (Penn State/NCAR), NMM (National Center for Environmental Prediction), and WRF (NCAR). The only models in the literature not based on the FD method are the FV models found in [7] and [1], and the SE and DG models presented in our paper. One of the biggest advantages that SE and DG methods have over the FD method is that no terrain following coordinates of the type presented in [16] need to be included in the governing equations. Of course, the orography has to be accounted for in some manner but element-based Galerkin (EBG) methods, such as FE, SE, FV, and DG, incorporate the orography via the definition of the grid. EBG methods do not require either orthogonal grids or grids with specific directions (such as the I and J indices in FD models); EBG models are inherently unstructured and, while requiring additional data structures for bookkeeping, completely liberate the method from the grid. This freedom from the grid has major repercussions in the implementation of these methods on distributed-memory computers in that no halo is required which translates into truly local algorithms that require very little communication across processors; instead, the communication stencil consists of the perimeter values of each processor (see [21]). The advantage that SE and DG methods have over the FD and FV methods is that high-order solutions (greater than fourth order) can be constructed quite naturally within the framework – such high-order properties are desirable because they reduce the dispersion errors associated with the discrete spatial operators [19]. In fact, the SE and DG formulations proposed in