Journal of Computational Physics 229 (2010) 4017–4032

Contents lists available at ScienceDirect

Journal of Computational Physics

journal homepage: www.elsevier.com/locate/jcp

A monotone finite volume method for advection–diffusion equations on unstructured polygonal meshes

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ARTICLE INFO

Article history:
Received 12 May 2009
Received in revised form 22 January 2010
Accepted 26 January 2010
Available online 4 February 2010

Keywords:
Advection–diffusion equation
Finite volume method
Discrete maximum principle
Monotone method
Unstructured mesh
Polygonal mesh

ABSTRACT

We present a new second-order accurate monotone finite volume (FV) method for the steady-state advection–diffusion equation. The method uses a nonlinear approximation for both diffusive and advective fluxes and guarantees solution non-negativity. The interpolation-free approximation of the diffusive flux uses the nonlinear two-point stencil proposed in Lipnikov [23]. Approximation of the advective flux is based on the second-order upwind method with a specially designed minimal nonlinear correction. The second-order convergence rate and monotonicity are verified with numerical experiments.

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1. Introduction

Accurate and reliable discretization methods inherit or mimic fundamental properties of continuous systems. The maximum principle and local mass conservation are the essential properties of the steady-state advection–diffusion equation. Despite a relative simplicity of this equation, an accurate discretization method that satisfies the discrete maximum principle (DMP) is hard to develop. Therefore, our focus is on a simplified version of the DMP that provides only solution non-negativity as is referred to as the monotonicity condition. Some physical quantities, such as concentration and temperature, are non-negative by their nature and their approximations should be non-negative as well. We develop a nonlinear finite volume (FV) method that satisfies the monotonicity condition for both diffusion-dominated and advection dominated regimes.

In advection dominated problems, a solution may have internal shocks and exponential or parabolic boundary layers. The thickness of these features is usually small compared to the mesh size and hence they cannot be resolved properly. In diffusion-dominated problems and highly anisotropic media, some of the diffusive fluxes may be poorly approximated if mesh cells are not aligned with the principle directions of the diffusive tensor. In both regimes, unwanted spurious (non-physical) oscillations may appear in the numerical solution. The design of advanced discretization methods that eliminate or significantly reduce these oscillations remains the field of extensive research for the last five decades.

One of the most popular finite element (FE) methods was developed by Brooks and Hughes in [6] and is referred to as the streamline upwind Petrov Galerkin (SUPG) method. The stabilization procedure proposed in this method improves significantly robustness of the FE discretization; however, the spurious oscillations around sharp layers may still appear in the numerical solution. Indeed, the SUPG method is neither monotone nor a monotonicity preserving method. Several

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0021-9991/$ - see front matter Published by Elsevier Inc.
doi:10.1016/j.jcp.2010.01.035

modifications and improvements of the SUPG method are reviewed in [14]. These modifications aim to design methods that satisfy the DMP, at least in some model cases, and were dubbed in [14] as the *spurious oscillations at layers diminishing* (SOLD) methods. Recently, another approach towards a robust FE method was developed in [17,18] and was dubbed as the *algebraic flux correction* method. The drawback of many FE methods is that they are formally not locally conservative on the original computational mesh, the property which is very desirable when a nonlinear advection–diffusion equation is coupled with other transport equations.

The finite volume (FV) methods guarantee the local mass conservation by construction. Recently, many new FV methods have been developed for the advection–diffusion equation (see [36,4,5,11,25,20] and references therein). It turns out that in the design of a monotone second-order accurate method, the approximation of diffusive fluxes is as challenging as that of the advective fluxes. The advective fluxes can be approximated via the upwinding approach [2] and controlled with different slope-limiting techniques [8,21,5] or introduction of an artificial viscosity [3,25]. For a long time, it was not clear how to approximate and control the diffusive fluxes in the case of general meshes and diffusion tensors. The theoretical analysis of DMP in the FE methods [9,16,35] imposes severe restrictions on the coefficients and computational mesh that are often violated in real-life simulations where the media is heterogeneous and anisotropic and the computational mesh may be strongly perturbed. In such a case, many advanced *linear* methods fail to satisfy the monotonicity condition [1,27,**?**]. This includes the mixed finite element (MFE), mimetic finite difference (MFD), and multi-point flux approximation (MPFA) methods that are locally conservative and second-order accurate on unstructured meshes. The linear two-point flux approximation FV method, still used in modeling flows in porous media, is monotone but not even first-order accurate for anisotropic problems. It was noticed in [5,14] that *nonlinear* approximations is the key ingredient and the price to pay for construction of a monotone and second-order accurate discretization. In [7] a nonlinear method has been developed for the Poisson equation. For a general diffusion equation, a number of nonlinear methods have been developed [10,15,19,28,22,29,34,37]. The optimization procedures were developed in [24,26] for tensorial diffusion equation.

In this article, the approximation of diffusive fluxes is based on a nonlinear two-point flux approximation method [23]. The original idea was proposed by Le Potier in [28] for triangular meshes. It was further analyzed and extended to shape-regular polygonal meshes (but scalar diffusion coefficient) in [22] and to tetrahedral meshes in [15]. Yuan and Sheng [37] extended the method to a bigger class of polygonal meshes with star-shaped cells and full tensor coefficients. In [34], the nonlinear diffusive fluxes and the operator splitting method were used for solving the unsteady advection–diffusion equation. All the above methods, in addition to *primary* unknowns defined at mesh cells, require solution values at mesh vertices that must be interpolated from the primary unknowns. As shown in [22,37], the choice of the interpolation method affects the accuracy of the nonlinear FV method for problems with constant diffusion coefficients. The interpolation problem becomes even a more challenging task for problems with discontinuous coefficients [37]. A interpolation-free nonlinear FV method was developed in Lipnikov et al. [23]. The numerical experiments presented there demonstrate that the new method requires lesser number of nonlinear iterations compared to the methods using interpolation algorithms. The interpolation-free method was extended to polyhedral meshes in Danilov and Vassilevski [10]; however, interpolation of solution at mid-edge points may be still required in certain pathological cases. Finally, new results on the DMP were reported in the conference proceedings [29].

The approximation of advective fluxes follows ideas of the *monotonic upstream-centered scheme for conservation laws* (MUSCL) introduced in van Leer [32]. A piecewise linear discontinuous reconstruction of the FV solution on polygonal cells allows to build more accurate advective fluxes that are also *nonlinear*. In order to control monotonicity and robustness of the method, we use a new slope limiting technique. In each cell, we minimize deviation of the reconstructed linear function from given values at selected points subject to some monotonicity constraints. For each cell, majority of these points are centers of the closest neighboring cells, except a few special cases. Other limiting procedures more closely related to the proposed method are discussed in Hubbard [12]. The essential difference lies in the points where monotonicity constraints are imposed and in the norm used to measure the difference between the unlimited and limited linear functions. The methods in Hubbard [12] use the Cartesian distance between gradients of these functions. We use an analog of the discrete L^2-norm over the reconstruction area.

In this article, we prove non-negativity of the discrete solution and verify it with numerical experiments. The developed nonlinear FV method is exact for linear solutions; therefore, the second-order asymptotic convergence rate is expected for problems with smooth solutions. This rate is observed in our numerical experiments.

One of the goals of this article it to study impact of coupling of diffusive and advective fluxes on the iterative nonlinear solver which is the major computational overhead in the proposed method. To focus numerical analysis on this issue, we consider only continuous anisotropic diffusion tensors and refer for derivation of diffusive fluxes for discontinuous problems to [28,15,37,10,23]. We consider the Picard method and prove that each iterative approximation to the discrete solution is non-negative. This extends similar results for diffusion problems [28,23] to advection–diffusion problems. We found out that difference in various methods for approximation of advective fluxes, which may be subtle from viewpoint of numerical methods for hyperbolic problems, may become important for stability of the Picard method. For instance, our selection of the set of admissible gradients is driven by this stability issue.

The paper outline is as follows. In Section 2, we state the steady advection–diffusion problem. In Section 3, we describe the nonlinear finite volume scheme. In Section 4, we prove monotonicity of the proposed scheme. In Section 5, we present numerical analysis of the scheme using triangular, quadrilateral and polygonal meshes.