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An extension of the discrete variational method to nonuniform grids

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ABSTRACT

The discrete variational method is a method used to derive finite difference schemes that inherit the conservation/dissipation property of the original equations. Although this method has mainly been developed for uniform grids, we extend this method to multidimensional nonuniform meshes.

1. Introduction

For PDEs that enjoy the conservation/dissipation property, numerical schemes that inherit that property are often advantageous, in that the schemes are fairly stable and yield qualitatively better numerical solutions in practice. For example, the Cahn–Hilliard equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2}{\partial x^2}\left(pu + ru^3 + q\frac{\partial^2 u}{\partial x^2}\right), \quad (t,x) \in (0,\infty)\times(0,L), \tag{1}$$

has a dissipation property

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L \left(\frac{p}{2}u^2 + \frac{r}{4}u^4 - \frac{q}{2}\left(\frac{\partial u}{\partial x}\right)^2\right)\mathrm{d}x \leqslant 0, \quad t > 0 \tag{2}$$

under certain boundary conditions. Here $p,\ q$ and r are real parameters that satisfy $p < 0,\ q < 0$ and $r > 0$. Although the existence of the term related to the negative dispersion effect in this equation often makes naive numerical schemes unstable, some numerical schemes that are designed so that they inherit the dissipation property (2) are proved to be stable and convergent [5,8].

Recently, Furihata and Matsuo [7–9,14–17] have developed the so-called "discrete variational method" that *automatically* constructs conservative/dissipative finite difference schemes for a class of PDEs with the conservation/dissipation property

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that stems from a certain variational structure. Originally, Furihata considered two types of equations in his first paper [7]. The first is the class of equations with the form

$$\frac{\partial u}{\partial t} = (-1)^{s+1}\left(\frac{\partial}{\partial x}\right)^{2s}\frac{\delta G}{\delta u}, \quad s = 0,1,2,3,\ldots, \quad x \in [0,L], \tag{3}$$

where $\delta G/\delta u$ is the variational derivative, which is defined by

$$\frac{\delta G}{\delta u} = \frac{\partial G}{\partial u} - \frac{\partial}{\partial x}\frac{\partial G}{\partial u_x}. \tag{4}$$

This class of equations includes the heat equation and the Cahn–Hilliard equation. The second is the class of equations with the form

$$\frac{\partial u}{\partial t} = \left(\frac{\partial}{\partial x}\right)^{2s+1}\frac{\delta G}{\delta u}, \quad s = 0,1,2,3,\ldots, \quad x \in [0,L], \tag{5}$$

which includes the advection equation and the KdV equation. $G(u,u_x)$ denotes a certain energy functional, such as the Hamiltonian or free energy. The total energy of these equations is defined by

$$H(t) := \int_0^L G(u,u_x)\mathrm{d}x. \tag{6}$$

As is widely known, under certain boundary conditions, (3) has the dissipation property

$$\frac{\mathrm{d}H}{\mathrm{d}t} \leqslant 0 \tag{7}$$

and (5) has the conservation property

$$\frac{\mathrm{d}H}{\mathrm{d}t} = 0. \tag{8}$$

Furihata proposed a method to derive finite difference schemes for (3) and (5) that inherit these properties after discretization, and his method has been extended to many other equations [9,14,16,17].

Until recently, the discrete variational method has been developed on uniform meshes only. However, especially in multidimensional problems, the use of nonuniform meshes is of importance, because the restriction to uniform meshes forces the domains to be rectangles. Furthermore, even in one-dimensional cases, nonuniform meshes are often useful when solutions exhibit locally complicated behavior.

In this paper, we extend the discrete variational method to nonuniform meshes. The extension is based on the "mapping method", where the change of coordinates plays an important role. For this reason, in the process of extension, we also show that after the change of coordinates, it remains that the conservation/dissipation property is obtained from the variational structure of the original equation.[1]

One of the most successful methods in structure preserving methods for PDEs is the mimetic approach ([1–4,11,19] and references therein). In this approach, differential operators are discretized even on unstructured meshes in a coordinate-invariant way while preserving the mass conservation, theorems of vector and tensor calculus, and the cohomology groups. Although there seem to be many similarities between the mimetic methods and our method, they are different. For example, whilst the quantities of interest in our method are from the variational structure, those in the mimetic approach are principally from geometric aspects of equations.

This paper is organized as follows.

In Section 2, we consider simple one-dimensional cases to clarify the idea of the extension, which employs the mapping method. Therefore, we briefly review the idea of the mapping method firstly in Section 2.1 and derive the conservation/dissipation property from the variational structure after the change of coordinates. In Section 2.2, we introduce a summation by parts formula on one-dimensional nonuniform grids, since it plays a very important role in the discrete variational method, similar to that of the integration by parts in conventional variational calculus. By using that formula, we define the discrete variational derivatives in Section 2.3. The dissipative and conservative schemes are defined in Sections 2.4 and 2.5, respectively.

In Section 3, we show a conservative scheme for the KdV equation as an example and give a numerical example. In Section 4, we extend the discrete variational method to multidimensional nonuniform meshes. Although we consider two-dimensional cases only for convenience of notation, the same procedure can be applied to cases greater than two dimensions. Because the integration by parts is replaced by the Gauss theorem in multidimensional cases, we show the discrete analogue of

[1] The basic idea has already been published in a Japanese paper [20].