Available online at www.sciencedirect.com

SCIENCE DIRECT®

Journal of Computational Physics 213 (2006) 330–357

JOURNAL OF COMPUTATIONAL PHYSICS

www.elsevier.com/locate/jcp

High-order discontinuous Galerkin methods using an *hp*-multigrid approach

Cristian R. Nastase *, Dimitri J. Mavriplis

Department of Mechanical Engineering, University of Wyoming, 1000 E. University Avenue, Laramie, WY 82071-3295, United States

Received 13 April 2005; received in revised form 12 August 2005; accepted 15 August 2005
Available online 7 October 2005

Abstract

The goal of this paper is to investigate and develop a fast and robust algorithm for the solution of high-order accurate discontinuous Galerkin discretizations of non-linear systems of conservation laws on unstructured grids. Herein we present the development of a spectral *hp*-multigrid method, where the coarse "grid" levels are constructed by reducing the order (p) of approximation of the discretization using hierarchical basis functions (p-multigrid), together with the traditional (h-multigrid) approach of constructing coarser grids with fewer elements. On each level we employ variants of the element-Jacobi scheme, where the Jacobian entries associated with each element are treated implicitly (i.e., inverted directly) and all other entries are treated explicitly. The methodology is developed for the two-dimensional non-linear Euler equations on unstructured grids, using both non-linear (FAS) and linear (CGC) multigrid schemes. Results are presented for the channel flow over a bump and a uniform flow over a four element airfoil. Current results demonstrate convergence rates which are independent of both order of accuracy (p) of the discretization and level of mesh resolution (h).

PACS: 47.11.+j; 83.85.Pt; 47.15.Ki

Keywords: Computational fluid dynamics; Discontinuous Galerkin finite element methods; High order methods; Numerical methods; Multigrid methods; Compressible flow; Gas dynamics

1. Introduction

While most currently employed CFD algorithms are asymptotically second-order accurate in time and in space, the use of higher-order discretizations in both space and time offers a possible avenue for improving the predictive simulation capability for many applications. This is due to the fact that higher-order methods exhibit a faster asymptotic convergence rate in the discretization error than lower (second)-order methods. For example, with a fourth-order accurate spatial discretization, the error is reduced by a factor of $2^4 = 16$

* Corresponding author. Tel.: +1 307 7665424; fax: +1 307 7662695.
E-mail addresses: nastase@uwyo.edu (C.R. Nastase), mavripl@uwyo.edu (D.J. Mavriplis).

0021-9991/$ - see front matter
doi:10.1016/j.jcp.2005.08.022

each time the mesh resolution is doubled, while a second-order accurate method only achieves a $2^2 = 4$ reduction in error with each doubling of the mesh resolution. Since a doubling of mesh resolution in three dimensions entails an increase of overall work by a factor of $2^3 = 8$, achieving an arbitrarily prescribed error tolerance with second-order accurate methods in three dimensions can quickly become unfeasible.

Thus, for increasingly high accuracy levels, higher-order methods ultimately become the method of choice. Therefore, the expectation is that an efficient higher-order discretization may provide an alternate path for achieving high accuracy in a flow with a wide disparity of length scales at reduced cost, by avoiding the use of excessive grid resolution.

On the other hand, for levels of accuracy often associated with mean-flow engineering calculations, higher-order methods have proved to be excessively costly compared to simpler second-order accurate methods. Clearly, because of the different asymptotic nature of these methods, the cost comparison between methods is a strong function of the required levels of accuracy. Nevertheless, for many engineering type calculations, higher-order methods have been found to be non-competitive compared to the simpler second-order accurate methods.

While the formulation of discretization strategies for higher-order methods such as discontinuous Galerkin [1–7] and streamwise upwind Petrov–Galerkin [8] methods are now fairly well understood, the development of techniques for efficiently solving the discrete equations arising from these methods has generally been lagging. This is partly due to the complex structure of the discrete equations originating from fairly sophisticated discretization strategies, as well as the current application of higher-order methods to problems where simple explicit time-stepping schemes are thought to be adequate solution mechanisms, due to the close matching of spatial and temporal scales, such as acoustic phenomena.

The development of optimal, or near optimal solution strategies for higher-order discretizations, including steady-state solutions methodologies, and implicit time integration strategies, remains one of the key determining factors in devising higher-order methods which are not just competitive but superior to lower-order methods in overall accuracy and efficiency.

Recent work by the second author has examined the use of spectral multigrid methods, where convergence acceleration is achieved through the use of coarse levels constructed by reducing the order (p) of approximation of the discretization (as opposed to coarsening the mesh) for discontinuous Galerkin discretizations [9]. The idea of spectral multigrid was originally proposed by Ronquist and Patera [10], and has been pursued for the Euler and Navier–Stokes equations by Fidkowski et al. [11–13] with encouraging results. Implicit multi-level solution techniques for high-order discretizations have also been developed by Lottes and Fisher [14].

In this work, we extend the original spectral multigrid approach described in [9] to the two-dimensional steady-state Euler equations, and couple the spectral p-multigrid approach with a more traditional agglomeration h-multigrid method for unstructured meshes. The investigation of efficient smoothers to be used at each level of the multigrid algorithm is also pursued, and comparisons between linear and non-linear solver strategies are made as well. The overall goal is the development of a solution algorithm which delivers convergence rates which are independent of p (the order of accuracy of the discretization) and independent of h (the degree of mesh resolution), while minimizing the cost of each iteration.

The key ingredient in the p-multigrid approach is to employ a hierarchical basis set together with a modal method. This renders the multigrid inter-level operators almost trivial to implement. This approach is rather different than the nodal method presented in [15], where a non-hierarchical basis (i.e., nodal basis based on Lagrange polynomials) is employed and the multilevel process requires rather complicated grid transfer operators. Moreover, in our methodology the coarse-grids are known a priori and the multilevel methodology is obtained by using known subsets of the original matrix. That is, the coarse grids correspond to a modal expansion in a lower space. This is also different than the algebraic multigrid (AMG) method [16] where a "matrix-free" operator is employed without prior knowledge of the coarse-grids and the multilevel process is obtained from an algebraic standpoint.

Note that the "*hp*-" terminology is commonly used to denote adaptive spatial and polynomial resolutions. This is referred to as "*h*-" and "*p*-adaptivity". Although the current multigrid methodology is not applied adaptively, it does make use of p-coarsened and h-coarsened levels leading to our terminology of either "p-multigrid", "h-multigrid" or "hp-multigrid" for the combined algorithm.