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A Fourier method for the fractional diffusion equation describing sub-diffusion ☆

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Abstract

In this paper, a fractional partial differential equation (FPDE) describing sub-diffusion is considered. An implicit difference approximation scheme (IDAS) for solving a FPDE is presented. We propose a Fourier method for analyzing the stability and convergence of the IDAS, derive the global accuracy of the IDAS, and discuss the solvability. Finally, numerical examples are given to compare with the exact solution for the order of convergence, and simulate the fractional dynamical systems.

Keywords: Fractional diffusion equation; Sub-diffusion; Implicit difference approximation; Fourier method; Stability; Convergence

1. Introduction

Fractional diffusion equations have attracted in recent years a considerable interest both in mathematics and in applications. These equations contain derivatives of fractional order in space, time or space–time [1]. They were used in modelling of many physical and chemical processes and in engineering [2–4]. Such evolution equations imply a fractional Fick's law for the flux that accounts for spatial and temporal non-locality [5]. Fractional calculus provides a powerful instrument for the description of memory and hereditary properties of substances [4]. Fractional-order differential equations have been the subject of worldwide attention by many research groups. In particular, the focus of Gorenflo, Mainardi and their co-authors' works on fractional calculus modelling (both deterministic and stochastic) and the derivation of fundamental solutions of the time, space and space–time fractional diffusion equations. They also presented discrete random walk models [6,7] and found that the fundamental solution can be interpreted as a probability

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density evolving in time of a self-similar stochastic process that can be viewed as a generalised diffusion process. Benson et al. [8,9] used a fractional advection–dispersion equation to simulate transport processes with heavy tails and demonstrated the equivalence between these heavy-tailed motions and transport equations that use fractional-order derivatives. Already in 1986, Wyss [10] considered the time fractional diffusion equation and gave the solution in closed form in terms of Fox functions. Then in 1989, Schneider and Wyss [11] considered the time fractional diffusion and wave equations, and the corresponding Green functions were obtained in closed form for arbitrary space dimensions in terms of Fox functions and their properties were exhibited. However, an explicit representation of the Green functions for the problem in a half-space was difficult to determine, except in the special cases $\alpha = 1$ (i.e., the first-order time derivative) with arbitrary n, or $n = 1$ with arbitrary α (i.e., the fractional-order time derivative). Huang and Liu [12] considered the time-fractional diffusion equations in an n-dimensional whole-space and half-space. They investigated the explicit relationships between the problems in whole-space with the corresponding problems in half-space by the Fourier–Laplace transform.

Fractional kinetic equations have proved particularly useful in the context of anomalous slow diffusion (sub-diffusion) [1]. The theoretical justification for the fractional diffusion equation, together with the abundance of physical and biological experiments demonstrating the prevalence of anomalous sub-diffusion, has led to an intensive effort in recent years to find accurate and stable methods of solution that are also straightforward to implement [13]. It has been suggested that the probability density function (pdf) $u(x, t)$ that describes anomalous sub-diffusive particles follows the fractional diffusion equation [1,13,14]:

$$\frac{\partial u(x,t)}{\partial t} = {}_0D_t^{1-\gamma}\left[\frac{\partial^2 u(x,t)}{\partial x^2}\right] + f(x,t), \quad t \geqslant 0, \tag{1}$$

where ${}_0D_t^{1-\gamma}u$ $(0 < \gamma \leqslant 1)$ denotes the Riemann–Liouville fractional derivative of order $1-\gamma$ of the function $u(x, t)$:

$${}_0D_t^{1-\gamma}u(x,t) = \frac{1}{\Gamma(\gamma)}\frac{\partial}{\partial t}\int_0^t \frac{u(x,\tau)}{(t-\tau)^{1-\gamma}}\,\mathrm{d}\tau, \tag{2}$$

with $0 < \gamma < 1$. For $\gamma = 1$ one recovers the identity operator and for $\gamma = 0$ the ordinary first-order derivative.

Some numerical methods for solving the space or time, or time–space fractional partial differential equations have been proposed [15–24]. However, the stability and convergence of numerical methods for fractional partial differential equations are deserved further investigations.

In this paper, we consider the initial-boundary value problem of the fractional diffusion equation describing sub-diffusion (FDE-sub) [13,25]:

$$\frac{\partial u(x,t)}{\partial t} = {}_0D_t^{1-\gamma}\left[\frac{\partial^2 u(x,t)}{\partial x^2}\right] + f(x,t), \quad 0 < t \leqslant T, \; 0 < x < L, \tag{3}$$

$$u(0,t) = \varphi(t), \quad 0 \leqslant t \leqslant T, \tag{4}$$

$$u(L,t) = \psi(t), \quad 0 \leqslant t \leqslant T, \tag{5}$$

$$u(x,0) = w(x), \quad 0 \leqslant x \leqslant L, \tag{6}$$

where $0 < \gamma \leqslant 1$; $f(x,t)$, $\varphi(t)$, $\psi(t)$ and $w(x)$ are sufficiently smooth functions.

Langlands and Henry [13] have investigated this problem. They proposed an implicit numerical scheme (L1 approximation), and discussed the accuracy and stability of this scheme. However, the global accuracy of the implicit numerical scheme has not been derived and it is apparent that the unconditional stability for all γ in the range $0 < \gamma \leqslant 1$ has not been established. The main purpose of this paper is to solve this problem via Fourier method.

The structure of the paper is as follows. In Section 2, we present an implicit difference approximation scheme. Sections 3 and 4 investigate the stability and convergence of the IDAS, respectively, using Fourier method. We prove that the IDAS is unconditionally stable for all γ in the range $0 < \gamma \leqslant 1$, derive the global accuracy of the IDAS, analyze the convergence of the IDAS , and discuss the solvability. Finally, some numerical examples are provided.