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Test models for improving filtering with model errors through stochastic parameter estimation

B. Gershgorin [a], J. Harlim [b,*], A.J. Majda [a]

[a] Department of Mathematics and Center for Atmosphere and Ocean Science, Courant Institute of Mathematical Sciences, New York University, NY 10012, United States

[b] Department of Mathematics, North Carolina State University, NC 27695, United States

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ABSTRACT

The filtering skill for turbulent signals from nature is often limited by model errors created by utilizing an imperfect model for filtering. Updating the parameters in the imperfect model through stochastic parameter estimation is one way to increase filtering skill and model performance. Here a suite of stringent test models for filtering with stochastic parameter estimation is developed based on the Stochastic Parameterization Extended Kalman Filter (SPEKF). These new SPEKF-algorithms systematically correct both multiplicative and additive biases and involve exact formulas for propagating the mean and covariance including the parameters in the test model. A comprehensive study is presented of robust parameter regimes for increasing filtering skill through stochastic parameter estimation for turbulent signals as the observation time and observation noise are varied and even when the forcing is incorrectly specified. The results here provide useful guidelines for filtering turbulent signals in more complex systems with significant model errors.

1. Introduction

Filtering is the process of obtaining the best statistical estimate of a natural system from partial observations of the signal from nature. In many contemporary applications in science and engineering, real time filtering of a turbulent signal from nature involving many degrees of freedom is needed to make accurate predictions of the future state. This is obviously a problem with significant practical impact. Important contemporary examples involve the real time filtering and prediction of weather and climate as well as the spread of hazardous plumes or pollutants. A major difficulty in accurate filtering of noisy turbulent signals with many degrees of freedom is model error [1]; the fact that the signal from nature is processed through an imperfect model where important physical processes are parameterized due to inadequate numerical resolution or incomplete physical understanding. Under these circumstances it is natural to devise strategies for parameter estimation to cope with model errors to improve filtering skill with model errors [2–9].

The simplest contemporary strategy to cope with model errors for filtering with an imperfect model nonlinear dynamical system depending on parameters, λ,

$$\frac{du}{dt} = F(u, \lambda) \tag{1}$$

is to augment the state variable u, by the parameters λ, and adjoin an approximate dynamical equation for the parameters

* Corresponding author.
E-mail address: jharlim@ncsu.edu (J. Harlim).

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$$\frac{d\lambda}{dt} = g(\lambda). \tag{2}$$

The right hand side of (2) is often chosen on an ad-hoc basis as $g(\lambda) \equiv 0$ or white noise forcing with a small variance [10,11]. The partial observations of the signal from nature are often processed by an Extended Kalman Filter (EKF, see [12–14]) applied to the augmented system in (1) and (2) where the parameters λ are estimated adaptively from these partial observations. Note that even if the original model in (1) is linear, it readily can have nonlinear dependence on the parameters λ so typically an EKF involving the linear tangent approximation and Kalman filtering is needed for parameter estimation in this standard case. Some recent applications of these and similar ideas to complex nonlinear dynamical system can be found in [2–4,6,7].

The topic of the present paper is the development of stringent test models for filtering turbulent signals from nature in the presence of significant model error and the improvement of filtering and skill through systematic stochastic parameter estimation. Here we develop a suite of exact Stochastic Parameterization Extended Kalman Filters (SPEKF) for stochastic parameter estimation and filtering in these test models following recent work of two of the authors [15,16] for filtering slow–fast systems. The test models include both additive and multiplicative bias corrections and their exactly solvable features yield important new guidelines for stochastic parameter estimation. Results below include comprehensive understanding of robust regimes for improved filtering skill with stochastic parameter estimation as well as delineating regimes of parameters with poor skill as different aspects of the observation time, observation noise variance, and the properties of the prototype signal from nature are varied. The exact statistical formulas with exponential growth in time developed in Section 2 below also point toward the potential lack of skill of EKF for stochastic parameter estimation.

1.1. Overview of the test models

In the test models here, the signals from nature are assumed to be given by the solution of the time dependent complex scalar Langevin equation

$$\frac{du(t)}{dt} = -\gamma(t)u(t) + \mathrm{i}\omega u(t) + \sigma\dot{W}(t) + f(t), \tag{3}$$

where $\dot{W}(t)$ is complex white noise and $f(t)$ is a prescribed external forcing. To generate significant model error as well as to mimic intermittent chaotic instability as often occurs in nature, we allow $\gamma(t)$ to switch between stable ($\gamma > 0$) and unstable ($\gamma < 0$) regimes according to a two-state Markov jump process. Here we regard $u(t)$ as representing one of the modes from nature in a turbulent signal as is often done in turbulence models [17–20], and the switching process can mimic physical features such as intermittent baroclinic instability [21]. As often occurs in practice, we assume that the switching process details are not known and only averaged properties are modeled. Thus, the Mean Stochastic Model (MSM) with significant model error given by

$$\frac{du(t)}{dt} = -\bar{\gamma}u(t) + \mathrm{i}\omega u(t) + \sigma\dot{W}(t) + \tilde{f}(t) \tag{4}$$

is utilized for filtering; here $\bar{\gamma} > 0$ is an average damping constant and $\tilde{f}(t)$ is possibly an incorrectly specified forcing. The SPEKF filters for stochastic parameter estimation are developed below in the context of true signal arising from (3) with the basic imperfect models developed in (4). The context of (3) and (4) provides a stringent test problem for improving filtering skill through stochastic parameter estimation which we develop below. In Section 2, we introduce the family of stochastic parameter estimation models and develop exactly solvable first and second order statistics for these models following [15,16]. Details of the model in (3) for true signal are described in Section 3 while a comprehensive study of the filtering skill through stochastic parameter estimation is presented in Section 4. In particular, Section 4 includes discussion of robustness and sensitivity to stochastic parameters in both the forced and unforced cases as well as learning the forcing from the filter process when the forcing is specified incorrectly. Section 5 contains concluding discussion, which indicates the fashion in which the stochastic estimation models, developed here, might be directly applied to turbulent dynamical systems with many degrees of freedom [22–25] as developed in a companion paper [26].

2. Exactly solvable test models for stochastic parameter estimation

2.1. Combined model

We consider a stochastic model for the evolution of state variable $u(t)$ together with *combined* additive, $b(t)$, and multiplicative, $\gamma(t)$, bias correction terms:

$$\begin{aligned}
\frac{du(t)}{dt} &= (-\gamma(t) + \mathrm{i}\omega)u(t) + b(t) + f(t) + \sigma\dot{W}(t), \\
\frac{db(t)}{dt} &= (-\gamma_b + \mathrm{i}\omega_b)(b(t) - \hat{b}) + \sigma_b\dot{W}_b(t), \\
\frac{d\gamma(t)}{dt} &= -d_\gamma(\gamma(t) - \hat{\gamma}) + \sigma_\gamma\dot{W}_\gamma(t)
\end{aligned} \tag{5}$$