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Journal of Computational Physics 211 (2006) 99–128
JOURNAL OF COMPUTATIONAL PHYSICS
www.elsevier.com/locate/jcp

Solving the 3D MHD equilibrium equations in toroidal geometry by Newton's method

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Received 11 August 2004; received in revised form 22 April 2005; accepted 18 May 2005
Available online 11 July 2005

Abstract

We describe a novel form of Newton's method for computing 3D MHD equilibria. The method has been implemented as an extension to the hybrid spectral/finite-difference Princeton Iterative Equilibrium Solver (PIES) which normally uses Picard iteration on the full nonlinear MHD equilibrium equations. Computing the Newton functional derivative numerically is not feasible in a code of this type but we are able to do the calculation analytically in magnetic coordinates by considering the response of the plasma's Pfirsch–Schlüter currents to small changes in the magnetic field. Results demonstrate a significant advantage over Picard iteration in many cases, including simple finite-β stellarator equilibria. The method shows promise in cases that are difficult for Picard iteration, although it is sensitive to resolution and imperfections in the magnetic coordinates, and further work is required to adapt it to the presence of magnetic islands and stochastic regions.

PACS: 52.30.Cv; 52.65.Kj

Keywords: MHD equilibrium; Newton's method; Princeton Iterative Equilibrium Solver; PIES

1. Introduction

Magnetohydrodynamics (MHD) is the basic single-fluid model of macroscopic plasma behavior [1,2]. It describes the effect of magnetic geometry on a plasma's macroscopic equilibrium and stability through the interaction of inertial, pressure, and magnetic forces. When a plasma evolves on timescales much longer

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0021-9991/$ - see front matter
doi:10.1016/j.jcp.2005.05.007

than the Alfvén time (typically microseconds in a fusion plasma) its behavior is governed by the MHD equilibrium equations,

$$\mathbf{J} \times \mathbf{B} = \nabla P, \tag{1}$$

$$\nabla \times \mathbf{B} = \mathbf{J}, \tag{2}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{3}$$

where P is the scalar pressure, $\mathbf{B}$ is the magnetic field, and $\mathbf{J}$ is the current density. Eq. (1) expresses equilibrium force balance in the plasma and Eq. (2) is Ampere's Law (displacement current is neglected in MHD). Toroidal MHD equilibria are important in modeling magnetic confinement devices for the nuclear fusion program (tokamaks and stellarators) because steady state reactor conditions are desirable. The equilibrium equations are analytically intractable for most realistic plasma configurations, so they must generally be solved by numerical techniques. Newton's method has been applied to simple one- and two-dimensional systems [3,4], confirming its well-known speed advantage and the ability to find some equilibria that raw Picard iteration cannot, but it has never been applied successfully in the general 3D case.

Our Newton equilibrium solver has been implemented in the hybrid spectral finite-difference Princeton Iterative Equilibrium Solver (PIES) [5] which normally uses direct (Picard) iteration of the full 3D equilibrium equations. PIES can be used to study realistic systems with magnetic islands and stochastic regions because it makes no assumptions about magnetic field structure, but a lot of solution blending between successive iterations can be required to get convergence. Newton's method is potentially a much faster direct algorithm, but computing the Newton functional derivative numerically is not feasible in a code of this type, for reasons given in Section 3. By considering the response of the plasma's pressure-driven Pfirsch–Schlüter currents to small magnetic perturbations, however, we are able to derive the functional derivative analytically in magnetic coordinates. Moreover, our version of Newton's method can be seen as a natural extension of the existing Picard formulation of PIES, as discussed in Section 3. We note that the Jacobian-free Newton–Krylov methods [6] are a potentially interesting alternative to the approach taken here.

Section 2 gives an overview of the Picard algorithm for computing 3D MHD equilibria, and its implementation in PIES. Section 3 shows how Newton's method can be formulated to solve the same problem, and how it relates to the Picard scheme. Section 4 gives essential details of our numerical implementation of Newton's method: the radial discretization and how its accuracy is affected by gauge choice for the vector potential, derivation of near origin radial dependence for Fourier coefficients, boundary conditions, and gross structure of the discretized system. Section 5 describes our analytic derivation of the Newton gradient term, and Section 6 gives some results on the performance of the Newton code. Finally, Section 7 summarizes and suggests further research.

2. Overview of the PIES code

The PIES code solves the MHD equilibrium equations in the form

$$\nabla \times \mathbf{B} = \mathbf{J}(\mathbf{B}), \tag{4}$$

where $\mathbf{J}$ is a complicated, nonlinear function of $\mathbf{B}$. Given $\mathbf{B}$, the component of $\mathbf{J}$ perpendicular to $\mathbf{B}$ is determined by the force balance equation, Eq. (1). The component parallel to $\mathbf{B}$, $\mathbf{J} \cdot \mathbf{B}$, is determined by $\nabla \cdot \mathbf{J} = 0$. As described below in this section, the PIES code solves Eq. (4) using Picard iteration,

$$\nabla \times \mathbf{B}_i = \mathbf{J}(\mathbf{B}_{i-1}), \tag{5}$$

where the subscript i denotes iteration number. The remaining sections of this paper will be concerned with the solution of Eq. (4) using Newton's method.