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A high-order and unconditionally stable scheme for the modified anomalous fractional sub-diffusion equation with a nonlinear source term

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ABSTRACT

The aim of this paper is to study the high order difference scheme for the solution of modified anomalous fractional sub-diffusion equation. The time fractional derivative is described in the Riemann–Liouville sense. In the proposed scheme we discretize the space derivative with a fourth-order compact scheme and use the Grünwald–Letnikov discretization of the Riemann–Liouville derivative to obtain a fully discrete implicit scheme. We analyze the solvability, stability and convergence of the proposed scheme using the Fourier method. The convergence order of method is $\mathcal{O}(\tau + h^4)$. Numerical examples demonstrate the theoretical results and high accuracy of the proposed scheme.

1. Introduction

In recent years there has been a growing interest in the field of fractional calculus [11,19,21,23]. Fractional differential equations have attracted increasing attention because they have applications in various fields of science and engineering [8]. Many phenomena in fluid mechanics, viscoelasticity, chemistry, physics, finance and other sciences can be described very successfully by models using mathematical tools from fractional calculus, i.e., the theory of derivatives and integrals of fractional order [9]. Some of the most applications are given in the book of Oldham and Spanier [22], the book of Podlubny [23] and the papers of Metzler and Klafter [18], Bagley and Trovik [1]. Many considerable works on the theoretical analysis [10,29] have been carried on, but analytic solutions of most fractional differential equations can not be obtained explicitly [9,13,24–26]. So many authors have resorted to numerical solution strategies based on convergence and stability analysis [2,3,8,12,28,30,31]. Liu et al. have carried on so many works on the finite difference method of fractional differential equations [14–16]. Sun et al. in [11,27] did nice investigation on the fractional diffusion-wave system and constructed some difference schemes with stability and convergence analysis. Also we refer the interested reader to [6,7] for some recent advances on fractional two-dimensional anomalous subdiffusion equation.

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There are several definitions of a fractional derivative of order $\alpha > 0$ [21,22]. The two most commonly used are the Riemann–Liouville and Caputo. The difference between the two definitions is in the order of evaluation [20]. Recently, models have been proposed to describe processes that become less anomalous as time progresses by the inclusion of a secondary fractional time derivative acting on a diffusion operator with a nonlinear source term [16,17]

$$\frac{\partial u(x,t)}{\partial t} = \left(\mathcal{A}\frac{\partial^{1-\alpha}}{\partial t^{1-\alpha}} + \mathcal{B}\frac{\partial^{1-\beta}}{\partial t^{1-\beta}}\right)\left[\frac{\partial^2 u(x,t)}{\partial x^2}\right] + f(u(x,t),x,t), \quad 0 < x < L,\ 0 < t \leqslant T, \tag{1.1}$$

$$\begin{aligned} u(0,t) &= \varphi_1(t), \quad 0 \leqslant t \leqslant T, \\ u(L,t) &= \varphi_2(t), \quad 0 \leqslant t \leqslant T, \end{aligned} \tag{1.2}$$

condition

$$u(x,0) = \psi(x), \quad 0 \leqslant x \leqslant L, \tag{1.3}$$

where $0 < \alpha, \beta \leqslant 1, \mathcal{A}, \mathcal{B} \geqslant 0$ and the nonlinear source term $f(u(x,t),x,t) \in C^2[0,L]$. The symbols $\frac{\partial^{1-\alpha}}{\partial t^{1-\alpha}}$ and $\frac{\partial^{1-\beta}}{\partial t^{1-\beta}}$ are the Riemann–Liouville fractional derivative operator and are defined as

$$\frac{\partial^{1-\alpha} u(x,t)}{\partial t^{1-\alpha}} = \frac{1}{\Gamma(\alpha)}\frac{\partial}{\partial t}\int_0^t \frac{u(x,\eta)}{(t-\eta)^{1-\alpha}}d\eta, \quad \frac{\partial^{1-\beta} u(x,t)}{\partial t^{1-\beta}} = \frac{1}{\Gamma(\beta)}\frac{\partial}{\partial t}\int_0^t \frac{u(x,\eta)}{(t-\eta)^{1-\beta}}d\eta,$$

where $\Gamma(\cdot)$ is the gamma function. Also, let $f(u,x,t)$ satisfies the Lipschitz condition with respect to u:

$$\left|f(\overline{u},x,t) - f(\widetilde{u},x,t)\right| \leqslant \mathcal{L}\left|\overline{u} - \widetilde{u}\right|, \quad \forall\, \overline{u}, \widetilde{u},$$

where $\mathcal{L}$ is a Lipschitz constant. Liu et al. [17] proposed a semi-discrete approximation and a full discrete finite element approximation for the modified anomalous subdiffusion (1.1)–(1.3) in a finite domain. They proved the stability and convergence of the proposed methods. Authors of [16] proposed a conditionally stable difference scheme for the solution of (1.1)–(1.3). They showed that the convergence order of method is $\mathcal{O}(\tau + h^2)$ with the energy method.

The aim of this paper is to propose an unconditionally stable difference scheme of order $\mathcal{O}(\tau + h^4)$ for the solution of Eq. (1.1). We apply a fourth-order difference scheme for discretizing the spatial derivative and Grünwald–Letnikov discretization for the Riemann–Liouville fractional derivative. We will discuss the stability of the proposed method by the Fourier method and show that the compact finite difference scheme converges with the spatial accuracy of fourth-order using Fourier analysis.

The outline of this paper is as follows. In Section 2, we introduce the derivation of the new method for the solution of Eq. (1.1). This scheme is based on approximating the time derivative of the mentioned equation by a scheme of order $\mathcal{O}(\tau)$ and the spatial derivative with a fourth-order compact finite difference scheme. In this section we obtain the matrix form of the proposed method and show its solvability. In Section 3 we prove the unconditional stability property of the method using the Fourier method. In Section 4 we present the convergence of method and show that the convergence order is $\mathcal{O}(\tau + h^4)$. The numerical experiments of solving Eq. (1.1) with the method developed in this paper for several test problems and comparison of numerical results with the results of some numerical methods in the literature are given in Section 5. Finally concluding remarks are drawn in Section 6.

2. Derivation of method

For positive integer numbers M and N, let $h = \frac{L}{M}$ denotes the step size of spatial variable, x, and $\tau = \frac{T}{N}$ denotes the step size of time variable, t. So we define

$$x_j = jh, \quad j = 0,1,2,\ldots,M,$$

$$t_k = k\tau, \quad k = 0,1,2,\ldots,N.$$

The exact and approximate solutions at the point (x_j, t_k) are denoted by u_j^k and U_j^k, respectively. We first state the fourth-order compact scheme of the second derivative in the following lemma which is taken from [8].

Lemma 1 [8]. *The fourth-order compact difference operator with maintaining three-point stencil to approximate the u_{xx} is*

$$\frac{\delta_x^2}{h^2\left(1 + \frac{1}{12}\delta_x^2\right)}u_j^k = \left.\frac{\partial^2 u}{\partial x^2}\right|_j^k - \frac{1}{240}\left.\frac{\partial^4 u}{\partial x^4}\right|_j^k h^4 + \mathcal{O}(h^6), \tag{2.1}$$

in which $\delta_x^2 u_j = (u_{j-1} - 2u_j + u_{j+1})$. Now using the relationship between the Grünwald–Letnikov formula and the Riemann–Liouville fractional derivative, we can write [30]