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Mixed mimetic spectral element method for Stokes flow: A pointwise divergence-free solution ☆

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ABSTRACT

In this paper we apply the recently developed mimetic discretization method to the mixed formulation of the Stokes problem in terms of vorticity, velocity and pressure. The mimetic discretization presented in this paper and in Kreeft et al. [51] is a higher-order method for curvilinear quadrilaterals and hexahedrals. Fundamental is the underlying structure of oriented geometric objects, the relation between these objects through the boundary operator and how this defines the exterior derivative, representing the grad, curl and div, through the generalized Stokes theorem. The mimetic method presented here uses the language of differential *k*-forms with *k*-cochains as their discrete counterpart, and the relations between them in terms of the mimetic operators: reduction, reconstruction and projection. The reconstruction consists of the recently developed mimetic spectral interpolation functions. The most important result of the mimetic framework is the commutation between differentiation at the continuous level with that on the finite dimensional and discrete level. As a result operators like gradient, curl and divergence are discretized exactly. For Stokes flow, this implies a pointwise divergence-free solution. This is confirmed using a set of test cases on both Cartesian and curvilinear meshes. It will be shown that the method converges optimally for all admissible boundary conditions.

1. Introduction

We consider Stokes flow, which models a viscous, incompressible fluid flow in which the inertial forces are negligible with respect to the viscous forces, i.e. when the Reynolds number is very small, $Re \ll 1$. Since $Re = UL/\nu$, small Reynolds numbers appear when either considering extremely small length scales, when dealing with a very viscous liquid or when one treats slow flows. Despite the simple appearance of the Stokes flow model, there exists a large number of numerical methods to simulate Stokes flow. They all reduce to two classes of either circumventing the Ladyshenskaya–Babuška–Brezzi (LBB) stability condition or satisfying this condition [36]. The first class can roughly be split into two subclasses, one is the group of stabilized methods, see e.g. [11,44] and the references therein, the other the group of least-squares methods, see e.g. [13,47].

The class that satisfies the LBB condition is the group of compatible methods. In compatible methods discrete vector spaces are constructed such that they satisfy the discrete LBB condition. Best known are the curl conforming Nédélec [56] and divergence conforming Raviart–Thomas [65] and Brezzi-Douglas-Marini [21] spaces. A subclass of compatible methods consists of *mimetic methods*. Mimetic methods do not solely search for appropriate vector spaces, but try to mimic structures

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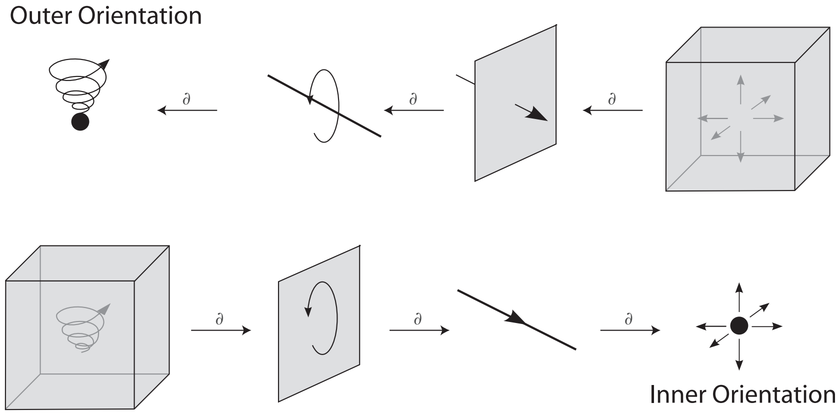

Fig. 1. The four geometric objects possible in $\mathbb{R}^3$, point, line, surface and volume, with outer- (above) and inner- (below) orientation. The boundary operator, ∂, maps k-dimensional objects to $(k-1)$-dimensional objects.

and symmetries of the continuous problem, see [14,20,51,53,58,71,72]. As a consequence of this mimicking, mimetic methods automatically preserve structures of the continuous formulation.

At the heart of the mimetic method we present is the generalized Stokes theorem, which couples the exterior derivative to the boundary operator. In vector calculus this theorem is equivalent to the classical Newton–Leibniz, Stokes circulation and Gauss divergence theorems. These well-known theorems relate the vector operators grad, curl and div to the restriction to the boundary of a manifold. Therefore, obeying geometry and orientation will result in satisfying exactly the mentioned theorems, and consequently performing the vector operators exactly in a finite dimensional setting. This is indeed what we are looking for and what our mimetic method has in common with finite volume methods, [42,71]. In a three dimensional space we distinguish between four types of submanifolds, that is, points, lines, surfaces and volumes, and two types of orientation, namely, outer- and inner-orientation. The inner and outer orientations can be seen as generalizations of the concept of tangential and normal in vector calculus, respectively. This geometric structure will form the backbone of the mimetic method to be discussed in this paper. It will reappear throughout the paper in various guises. Examples of submanifolds in $\mathbb{R}^3$ are shown in Fig. 1 together with the action of the boundary operator.

By creating a quadrilateral or hexahedral mesh, we divide the physical domain in a large number of these geometric objects, and to each geometric object we associate a discrete unknown. This implies that these discrete unknowns are *integral quantities*. Since the generalized Stokes theorem is an integral equation, it follows for example that taking a divergence in a volume is equivalent to taking the sum of the integral quantities associated to the surrounding surface elements, i.e. the fluxes. So using integral quantities as degrees of freedom to perform a vector operation like grad, curl or div, is equivalent to taking the sum of the degrees of freedom located at its boundary.

These relations are of purely topological nature, which follows directly by the duality between the exterior derivative and the boundary operator through the generalized Stokes theorem. The action of the boundary operator is indicated by the horizontal connections between the geometric objects in Fig. 1. They form a topological sequence or complex. This sequence is fundamental. It has a direct connection with the de Rham complex, chain complex and cochain complex, describing relations of the underlying PDEs, the computational mesh and the discretization of the PDEs.

In this work we use the language of differential geometry to identify these structures, since it clearly identifies the metric and metric-free part of the PDEs. The latter has a discrete counterpart in the language of algebraic topology. In mimetic methods we employ commuting diagrams to indicate the strong analogy between differential geometry and algebraic topology. The most important commuting property employed in this work is the commutation between the projection operator and differentiation in terms of the exterior derivative. This means that also in finite dimensional spaces, operations like gradient, curl and divergence are performed exactly. This implies, among others, and most importantly that incompressible Navier–Stokes and Stokes flow are guaranteed to be pointwise divergence-free, because the projection operator commutes with the divergence operator.

The similarities between differential geometry and algebraic topology in physical theories were first described by Tonti [72]. A mimetic framework relating differential forms and cochains was initiated by Hyman and Scovel [46], and extended first by Bochev and Hyman [14], and later by Kreeft et al. [51]. A framework, closely related to the mentioned mimetic framework, is the finite element exterior calculus framework by Arnold et al. [5,6]. A more geometric approach is described in the work by Desbrun et al. [29,30]. An excellent introduction and motivation for the use of differential forms in the description of physics and the use in numerical modeling can be found in the 'Japanese papers' by Bossavit, [15,16]. In Demkowicz et al. [28], *hp* finite element spaces were treated that are conforming to the de Rham diagram.

We make use of spectral element interpolation functions as basis functions. In the past nodal spectral elements were mostly used in combination with Galerkin (GSEM) [10,48], and least-squares formulations (LSSEM) [60,62]. The GSEM satisfies the LBB compatibility condition by lowering the polynomial degree of the pressure by two with respect to the velocity.