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Double-families of quasi-positive Darcy-flux approximations with highly anisotropic tensors on structured and unstructured grids

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ABSTRACT

This paper focuses on flux-continuous pressure equation approximation for strongly anisotropic media. Previous work on families of flux-continuous schemes for solving the general geometry–permeability tensor pressure equation has focused on single-parameter families. These schemes have been shown to remove the O(1) errors introduced by standard two-point flux reservoir simulation schemes when applied to full-tensor flow approximation. Improved convergence of the schemes has also been established for specific quadrature points. However these schemes have conditional M-matrices depending on the strength of the off-diagonal tensor coefficients. When applied to cases involving full-tensors arising from strongly anisotropic media, the point-wise continuous schemes can fail to satisfy the maximum principle and induce severe spurious oscillations in the numerical pressure solution.

New double-family flux-continuous locally conservative schemes are presented for the general geometry–permeability tensor pressure equation. The new double-family formulation is shown to expand on the current single-parameter range of existing conditional M-matrix schemes. The conditional M-matrix bounds on a double-family formulation are identified for both quadrilateral and triangle cell grids. A quasi-positive QM-matrix analysis is presented that classifies the behaviour of the new schemes with respect to double-family quadrature in regions beyond the M-matrix bounds. The extension to double-family quadrature is shown to be beneficial, resulting in novel optimal anisotropic quadrature schemes. The new methods are applied to strongly anisotropic full-tensor field problems and yield results with sharp resolution, with only minor or practically zero spurious oscillations.

1. Introduction

This paper presents the development of new double-families of flux-continuous finite-volume methods for the pressure equation resulting from Darcy's law and mass conservation. Key physical constraints of continuity in normal flux and full-pressure continuity are imposed at control-volume interfaces.

The derivation of algebraic flux continuity conditions for full-tensor discretization operators has lead to families of efficient locally conservative pointwise flux-continuous control-volume distributed (CVD) finite-volume schemes for determining the discrete pressure and velocity fields [1–5]. These schemes are constructed using linear triangular pressure support (TPS) inside each subcell for both quadrilateral and triangle grids, and are classified by the quadrature parameterization $0 < q \leqslant 1$. Schemes of this type are also called multi-point flux approximation schemes or MPFA [6] where focus has been on a scheme that is the default member of the above mentioned family with $q = 1$. Further schemes of this type are presented in [7,8] and via a novel mixed method [9]. Other schemes that preserve flux continuity have been developed from

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variational frameworks, using the mixed finite element method e.g. [10–15] and related work [16] and discontinuous galerkin methods [17,18], however these schemes require additional degrees of freedom.

When applying the flux-continuous schemes to the elliptic pressure equation with a strongly anisotropic full-tensor field they can fail to satisfy the maximum principle (as with other finite element and finite-volume methods) and result in spurious oscillations in the numerical pressure solution. M-matrix conditions were first derived in [19,1], monotone matrix conditions are presented in [20]. Grid optimization techniques have been proposed for improving stability of the discrete system for variable anisotropy [21]. Discretization schemes that help to improve stability for high anisotropy are presented in [4,22,23]. Non-linear methods have also been proposed, [24,25] (flux-splitting) and [26,27] (positivity preserving), both of which have been shown to yield numerical pressure solutions that are free of spurious oscillations.

In this paper new families of flux-continuous, locally conservative, finite-volume schemes are presented for solving the general-tensor pressure equation. The new schemes have full-pressure continuity imposed across control-volume faces, in contrast to the earlier families of flux-continuous schemes with pointwise continuity in pressure and flux. This work extends the single-parameter family of [22] to double-families of schemes on quadrilateral and triangular cell grids with general flexibility in quadrature that allow different quadrature points to be used on different control-volume subfaces.

For strongly anisotropic full-tensor cases where M-matrix conditions are violated, the earlier pointwise continuous families of schemes cannot avoid decoupling of the solution which leads to severe spurious oscillations in the discrete solution [22]. The new schemes minimiSe spurious oscillations in discrete pressure solutions. The new formulation leads to more robust quasi-positive families of flux-continuous schemes applicable to general discontinuous full-tensor fields. Particular focus is on two types of the many possible schemes within the formulation. The first involves identifying a family of optimal support schemes according to anisotropy, while the second involves extreme anisotropic quadrature schemes.

This paper is organized as follows: Section 2 gives a description of the single phase flow problem encountered in reservoir simulation with respect to the general-tensor pressure equation. Two-phase flow is also considered in the results section. The double-family of Full-Pressure Support (FPS) schemes is introduced in Section 3 for quadrilateral and triangular cell-vertex schemes. The relationship between FPS and control-volume finite element (CVFE) double-families is given in Section 4. Positivity is defined and double-family M-matrix conditions for quadrilateral schemes are derived in Section 5. Optimal and anisotropic quadrature schemes are introduced in Section 6, together with extreme anisotropic quadrature rules. Decoupled approximation, important consequences and implications for monotone schemes are presented in Section 7. Triangular grid M-matrix conditions are given in Section 8. Quasi-positive QM-matrices are defined in Section 9, where the double-family schemes are classified when M-matrix limits are exceeded. The cell-vertex FPS triangular grid formulation also leads to optimal schemes when anisotropy angle favoring triangulation is employed. Results for the new FPS schemes are presented in Section 10. Comparisons between the earlier TPS pointwise flux-continuous CVD(MPFA) schemes and new FPS schemes clearly demonstrate the benefits of the new schemes, both in terms of significantly reducing spurious oscillations and improving solution resolution. Conclusions follow in Section 11.

2. Flow equation and problem description

The analytical pressure equation is formulated in a general curvilinear coordinate system that is defined with respect to a uniform dimensionless transform space with a (ξ,η) coordinate system. (Neumann/Dirichlet) boundary conditions on boundary $\partial\Omega$. Choosing Ω to represent an arbitrary control-volume comprised of surfaces that are tangential to constant (ξ,η) respectively, where $\partial\Omega$ is the boundary of Ω and $\hat{n}$ is the unit outward normal. Spatial derivatives are expressed as $\phi_x = J(\phi,y)/J(x,y), \phi_y = J(x,\phi)/J(x,y)$ where $J(x,y) = x_\xi y_\eta - x_\eta y_\xi$ is the Jacobian. Resolving the x,y components of Darcy velocity $\mathbf{V} = -\mathbf{K}\nabla\phi$ along the unit normals to the curvilinear coordinates (ξ,η), e.g., normal to ξ = constant, $\hat{\mathbf{n}}ds = (y_\eta, -x_\eta)d\eta$ gives rise to the general-tensor Darcy-flux components

$$F = -\int (T_{11}\phi_\xi + T_{12}\phi_\eta)\,d\eta, G = -\int (T_{12}\phi_\xi + T_{22}\phi_\eta)\,d\xi, \tag{1}$$

where general tensor $\mathbf{T} = |J|\mathbf{J}^{-1}\mathbf{K}\mathbf{J}^{-T}$ elements are given by

$$\begin{aligned} T_{11} &= \left(K_{11}y_\eta^2 + K_{22}x_\eta^2 - 2K_{12}x_\eta y_\eta\right)/J, \\ T_{22} &= \left(K_{11}y_\xi^2 + K_{22}x_\xi^2 - 2K_{12}x_\xi y_\xi\right)/J, \\ T_{12} &= (K_{12}(x_\xi y_\eta + x_\eta y_\xi) - (K_{11}y_\eta y_\xi + K_{22}x_\eta x_\xi))/J \end{aligned} \tag{2}$$

and result from the Piola transformation. Matrix $\mathbf{K}$ is a diagonal or full elliptic cartesian tensor. The closed integral can be written as

$$\int\int_\Omega \frac{(\partial_\xi \widetilde{F} + \partial_\eta \widetilde{G})}{J} J d\xi d\eta = \triangle_\xi F + \triangle_\eta G = M \tag{3}$$

where $\triangle_\xi F$, $\triangle_\eta G$ are the differences in net flux with respect to ξ and η respectively, $\widetilde{F} = T_{11}\phi_\xi + T_{12}\phi_\eta$, and $\widetilde{G} = T_{12}\phi_\xi + T_{22}\phi_\eta$. M represents a specified flow rate. Ellipticity of $\mathbf{T}$ follows from that of K. full-tensors can arise from upscaling, unstructured grids and local orientation of the grid and permeability field.