Journal of Computational Physics 229 (2010) 660–680

Contents lists available at ScienceDirect

Journal of Computational Physics

journal homepage: www.elsevier.com/locate/jcp

On the role of Riemann solvers in Discontinuous Galerkin methods for magnetohydrodynamics

V. Wheatley [a,*], H. Kumar [b], P. Huguenot [b]

[a] Centre for Hypersonics, The University of Queensland, Australia
[b] Seminar for Applied Mathematics, ETH Zurich, Switzerland

ARTICLE INFO

Article history:
Received 3 March 2009
Received in revised form 2 September 2009
Accepted 4 October 2009
Available online 13 October 2009

Keywords:
Riemann solvers
Discontinuous Galerkin methods
Limiters

ABSTRACT

It has been claimed that the particular numerical flux used in Runge–Kutta Discontinuous Galerkin (RKDG) methods does not have a significant effect on the results of high-order simulations. We investigate this claim for the case of compressible ideal magnetohydrodynamics (MHD). We also address the role of limiting in RKDG methods.

For smooth nonlinear solutions, we find that the use of a more accurate Riemann solver in third-order simulations results in lower errors and more rapid convergence. However, in the corresponding fourth-order simulations we find that varying the Riemann solver has a negligible effect on the solutions.

In the vicinity of discontinuities, we find that high-order RKDG methods behave in a similar manner to the second-order method due to the use of a piecewise linear limiter. Thus, for solutions dominated by discontinuities, the choice of Riemann solver in a high-order method has similar significance to that in a second-order method. Our analysis of second-order methods indicates that the choice of Riemann solver is highly significant, with the more accurate Riemann solvers having the lowest computational effort required to obtain a given accuracy. This allows the error in fourth-order simulations of a discontinuous solution to be mitigated through the use of a more accurate Riemann solver.

We demonstrate the minmod limiter is unsuitable for use in a high-order RKDG method. It tends to restrict the polynomial order of the trial space, and hence the order of accuracy of the method, even when this is not needed to maintain the TVD property of the scheme.

1. Introduction

To simulate compressible flows that contain shocks along with small-scale features such as turbulence, we require numerical methods that are shock capturing, but also exhibit high-order accuracy and low numerical dissipation away from shocks [1]. Runge–Kutta Discontinuous Galerkin (RKDG) methods are shock capturing and high-order accurate away from discontinuities, thus they are a candidate method for carrying out such simulations.

Discontinuous Galerkin (DG) methods were first introduced by Hill and Reed [2] for the neutron transport equations (linear hyperbolic equations). LeSaint and Raviart [3] proved a rate of convergence of $O(\Delta x)^k$ for general triangulations and of $O(\Delta x)^{k+1}$ for Cartesian meshes, where Δx is the element size and k is the polynomial order of the approximate solution. In case of general triangulations, this result was then improved by Jhonson and Pitkaranta [4] to $O(\Delta x)^{k+1/2}$, which was confirmed to be optimal by Peterson [5].

* Corresponding author. Tel.: +61 7 33652969.
E-mail address: v.wheatley@uq.edu.au (V. Wheatley).

0021-9991/$ - see front matter © 2009 Elsevier Inc. All rights reserved.
doi:10.1016/j.jcp.2009.10.003

These methods were then generalized for systems of hyperbolic conservation laws by Cockburn and co-workers [6–10]. In space the solution is approximated using piecewise polynomials on each element. Exact or approximate Riemann solvers from finite volume methods are used to compute the numerical fluxes between elements. Limiters are used to achieve non-oscillatory approximate solutions, if they contain shocks [11]. For these reasons, DG methods can be seen as generalization of finite volume methods to higher order. For time integration, the total variation diminishing (TVD) explicit Runge–Kutta (RK) methods proposed by Shu and Osher [12] are used.

RKDG methods have many important advantages. Like finite element methods, RKDG methods are well suited for simulating flows in complicated geometries. These methods can easily handle adaptivity strategies, because of the assumed discontinuity of the solution at element interfaces. This allows refining or unrefining of the triangulation to be done without taking into account the continuity restrictions typical of conforming finite element methods. Similarly, the degree of the polynomial approximation within an element can be changed without affecting the solution on other elements. Another important advantage is that these methods are highly parallelizable because to update the solution on a given element, only information from elements with which it shares a face is needed.

At present, it is believed that the particular numerical flux or Riemann solver used does not have a significant effect on the results of high-order RKDG simulations [13]. Such a conclusion is supported by numerical evidence such as that shown in Fig. 1. This figure shows the results of simulations of the MHD shock tube problem described in Section 4.1 with three different flux calculators (see Section 4 for descriptions), along with the exact solution to the problem. The results of first- and second-order simulations are shown in Fig. 1(a) and (b), respectively. Comparing these results, it appears that the absolute error in the numerical solution is far less sensitive to the choice of numerical flux when the second-order scheme is used. This seems to indicate that as the order of a simulation increases, the choice of numerical flux becomes less significant. This view has lead to the simple and highly dissipative Lax–Friedrichs (LF) flux being used within many RKDG methods [13]. Our goal is to rigorously examine the effect of more accurate numerical flux calculators in high-order RKDG methods, with particular emphasis on high-order simulations featuring discontinuities. The influence of accurate flux calculators in high-order RKDG methods is intimately tied to the performance of the limiters in the method. For this reason, we also examine the performance of limiters in high-order simulations.

For the Euler equations, the effect of Riemann solvers has been previously evaluated by Qiu et al. [14]. One- and two-dimensional numerical simulations were carried out to compare various Riemann solvers based on performance measures such as numerical error, resolution of discontinuities and CPU times. The LF flux was shown to require the least CPU-time among all the fluxes that were compared, but it also produced the largest numerical errors. Whereas second-order fluxes such as Lax–Wendroff (LW) and Warming–Beam (WB) were found to be unstable. The Harten, Lax and van Leer (HLL) [15], HLLC [16] and MUSTA [17] fluxes were proposed as good choices for RKDG simulations. However, the data generated was not correlated to demonstrate which scheme is the most computationally efficient, or if the benefits of using more accurate Riemann solvers are dependent on the order of accuracy. In addition, error norms were only computed for a smooth linear problem, while we anticipate that the use of accurate Riemann solvers will be most significant in discontinuous nonlinear problems.

This report is organized as follows: in the next section we present the governing equations for the simulations. In Section 3 a brief description of the RKDG method is presented. In Section 4, a number of flux calculators for ideal MHD are tested, leading to the selection of appropriate flux calculators for use in the RKDG method in different circumstances. The limiters used within the RKDG method are described and tested in Section 5. The results of numerical test cases are presented and analyzed in Section 6. Finally, the conclusions that have been drawn from this work are presented in Section 7.

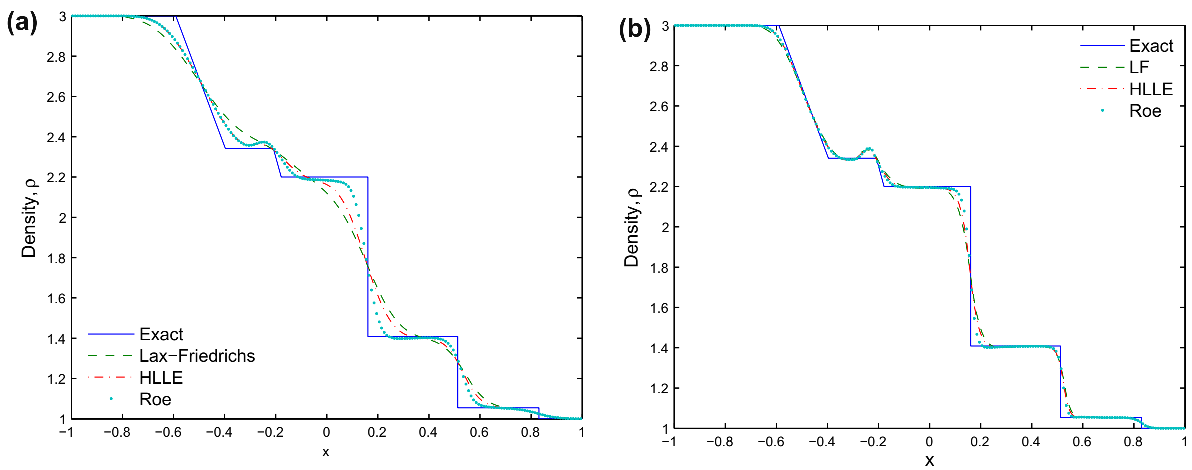

Fig. 1. Density profiles at $t = 0.4$ from (a) first-order and (b) second-order accurate simulations of the MHD shock tube problem described in Section 4.1. The exact solution to the problem is shown along with numerical results using the LF, HLLE and Roe fluxes described in Section 4.