Journal of Computational Physics 229 (2010) 828–850

Contents lists available at ScienceDirect

Journal of Computational Physics

journal homepage: www.elsevier.com/locate/jcp

Implicit LU-SGS algorithm for high-order methods on unstructured grid with *p*-multigrid strategy for solving the steady Navier–Stokes equations

M. Parsani [a,*], K. Van den Abeele [a], C. Lacor [a], E. Turkel [b]

[a] Vrije Universiteit Brussel, Department of Mechanical Engineering, Fluid Dynamics and Thermodynamics Research Group, Pleinlaan 2, 1050 Brussel, Belgium
[b] Tel-Aviv University, Department of Mathematics, Tel-Aviv, Ramat Aviv 69978, Israel

ARTICLE INFO

Article history:
Received 4 December 2008
Received in revised form 6 October 2009
Accepted 9 October 2009
Available online 17 October 2009

Keywords:
Navier–Stokes
High-order methods
Implicit LU-SGS algorithm
Von Neumann analysis
p-Multigrid

ABSTRACT

The fluid dynamic equations are discretized by a high-order spectral volume (SV) method on unstructured tetrahedral grids. We solve the steady state equations by advancing in time using a backward Euler (BE) scheme. To avoid the inversion of a large matrix we approximate BE by an implicit lower–upper symmetric Gauss–Seidel (LU-SGS) algorithm. The implicit method addresses the stiffness in the discrete Navier–Stokes equations associated with stretched meshes. The LU-SGS algorithm is then used as a smoother for a *p*-multigrid approach. A Von Neumann stability analysis is applied to the two-dimensional linear advection equation to determine its damping properties. The implicit LU-SGS scheme is used to solve the two-dimensional (2D) compressible laminar Navier–Stokes equations. We compute the solution of a laminar external flow over a cylinder and around an airfoil at low Mach number. We compare the convergence rates with explicit Runge–Kutta (E-RK) schemes employed as a smoother. The effects of the cell aspect ratio and the low Mach number on the convergence are investigated. With the *p*-multigrid method and the implicit smoother the computational time can be reduced by a factor of up to 5–10 compared with a well tuned E-RK scheme.

1. Introduction

Spatially high-order accurate numerical schemes are being developed for use in a variety of flow problems. In Computational Fluid Dynamics (CFD), they are being used for Direct Numerical Simulation (DNS), Large Eddy Simulation (LES), Computational Aeroacoustics (CAA), turbulent combustion etc. where the accurate resolution of small scales is required. In addition, since CFD is increasingly used as an industrial design and analysis tool, it requires unstructured grids for efficient meshing. High-order accuracy must therefore be achieved on unstructured grids. Discontinuous Galerkin (DG) schemes [1–5], Residual Distribution (RDS) [6] and the more recently developed Spectral Volume (SV) [7–15] and Spectral Difference (SD) [17–19] schemes are especially suited for these purposes.

However, when high-order schemes are combined with classical solution methods, such as explicit Runge–Kutta (E-RK) solvers, they suffer from a restrictive CFL condition and hence a relatively slow convergence rate. In addition to this, the solver must also be able to deal with the geometrical stiffness imposed by the Navier–Stokes grids where high-aspect ratios

* Corresponding author. Tel.: +32 2 6292399; fax: +32 2 6292880.
E-mail addresses: mparsani@vub.ac.be (M. Parsani), kvdabeel@vub.ac.be (K. Van den Abeele), chris.lacor@vub.ac.be (C. Lacor), turkel@post.tau.ac.il (E. Turkel).

0021-9991/$ - see front matter
doi:10.1016/j.jcp.2009.10.014

occur near walls. In the case of compressible solvers there is an additional stiffness when solving for low speed flows caused by the disparate eigenvalues of the system.

High-order schemes possess less numerical dissipation than the lower-order ones. Consequently, it takes an excessive amount of CPU-time to reach a steady-state solution with explicit solvers. Therefore, efficient solvers are needed to fully fulfill the potential of high-order methods. Implicit time-integration schemes can be used to deal with these problems. These schemes can advance the solution with significantly larger time-steps compared to explicit methods. However, they may be more expensive than explicit schemes if the algebraic solver employed is not efficient. Recently, there has been some research on implicit Runge–Kutta (I-RK) schemes [20,21]. In Bijl et al. [20], I-RK solvers were investigated in combination with a standard cell-centered finite volume scheme with artificial dissipation added for stability. It was observed that significant potential improvements in the temporal efficiency of implicit schemes could be achieved from algebraic solver developments. In [21] the convergence of an E-RK scheme with h-multigrid was accelerated by preconditioning with a fully implicit operator and the resulting RK/Implicit Residual scheme was used as a smoother for an h-multigrid algorithm. It was demonstrated that the implicit preconditioner reduced the computational time of a well tuned E-RK scheme by a factor between 4 and 10. Both studies [20,21], concluded that solver improvements can be more dramatic than improvements in integration techniques.

Another indispensable tool for efficiency is the multigrid algorithm. In the traditional h-multigrid approach, efficiency is achieved by switching to coarser spatial grids. In a p-multigrid algorithm, a high-order solution representation is transferred to a lower-order one and the multigrid algorithm uses iterations on sequences of different solution representations instead of different grids. Essential for efficiency is that the solver is a good smoother of high-frequency error components and this should also hold for high-aspect ratio grids.

In the present study we combine a SV discretization in space and an implicit lower–upper symmetric Gauss–Seidel (LU-SGS) in (pseudo) time. This is accelerated by a full p-multigrid strategy. The LU scheme was started by Jameson and Turkel [22] and later reformulated to use symmetric Gauss–Seidel by Jameson and Yoon in the context of second-order central schemes [23]. It was recently rediscovered by Sun et al. [24] and adapted for use with SD schemes. In Parsani et al. [27] it was coupled with the SV scheme and a full p-multigrid algorithm. Here, the LU-SGS algorithm with the backward Euler method is evaluated both with analysis and computation. The damping properties of the implicit method are evaluated with a Von Neumann stability analysis for a model 2D linear advection equation. In [25,26] this analysis was applied to implicit schemes on Cartesian grids for classical upwind and central schemes. In the present work the analysis is on triangular grids defined by a generating pattern, and for high-order SV schemes.

The implicit LU-SGS scheme is used to solve the two-dimensional steady laminar flows over a cylinder and a NACA0012 airfoil at low Mach number. For the two cases the order of the SV scheme is restricted to 2 because of the curved boundaries. Currently a first-order interpolation is used for the boundary shape. High-order schemes, would require a more accurate interpolation, especially on the relatively coarse grids that are being used in combination with high-order schemes [13].

The convergence behavior and the computational effort of the implicit LU-SGS algorithm is compared with that of a family of optimized E-RK smoothers used in Van den Abeele et al. [14]. The influence of the mesh aspect ratio and the low Mach number is investigated and the solutions are compared with experimental and numerical results found in the literature. For the NACA0012 airfoil test case the convergence behavior of the LU-SGS solver is compared with that of RK/Implicit Residual scheme used in [21]. The latter code is second-order accurate in space and uses a finite volume approach with quadrilateral cells on structured grids.

The remainder of this article is organized as follows. A brief summary of the SV method is given in section 2. In Section 3, the p-multigrid algorithm is described with the definition of a general restriction operator [27]. In Section 4, the explicit Runge–Kutta schemes (E-RK) are described. In Section 5, the implicit LU-SGS with backward Euler (BE) scheme is discussed. A Von Neumann stability analysis for the LU-SGS with backward Euler method and for a general SV schemes on triangular grids is described in Section 6. Section 7 shows the stability analysis' results for the second-order SV scheme. Section 8 deals with the numerical test cases, before finally drawing conclusions in Section 9.

2. Spectral volume method

The spectral volume (SV) method is used to solve conservation laws (1)

$$\frac{\partial \mathbf{W}}{\partial t} + \nabla \cdot \mathcal{F}(\mathbf{W}) = 0, \tag{1}$$

where $\mathbf{W}$ is the state vector of conservative variables and $\mathcal{F}(\mathbf{W})$ is the flux density tensor.

The computational domain V is divided in N^{SV} cells V_i, called spectral volumes, with volume $|V_i|$. Each SV is further subdivided into control volumes (CV) $V_{i,j}$. Integrating Eq. (1) over such a CV and applying the Gauss theorem gives

$$\frac{\partial}{\partial t}\left(\overline{W}_{i,j}|V_{i,j}|\right) = -\int_{\partial V_{i,j}} \mathcal{F} \cdot d\mathbf{s} = R_{i,j}, \tag{2}$$

where $|V_{i,j}|$ is the volume of $V_{i,j}, R_{i,j}$ is the residual corresponding to $V_{i,j}$ and $\overline{W}_{i,j}$ is the CV average defined by