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Analysis and algorithms for a regularized cauchy problem arising from a non-linear elliptic PDE for seismic velocity estimation

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ABSTRACT

In the present work we derive and study a non-linear elliptic PDE coming from the problem of estimation of sound speed inside the Earth. The physical setting of the PDE allows us to pose only a Cauchy problem, and hence is ill-posed. However, we are still able to solve it numerically on a long enough time interval to be of practical use. We used two approaches. The first approach is a finite difference time-marching numerical scheme inspired by the Lax–Friedrichs method. The key features of this scheme is the Lax–Friedrichs averaging and the wide stencil in space. The second approach is a spectral Chebyshev method with truncated series. We show that our schemes work because of (i) the special input corresponding to a positive finite seismic velocity, (ii) special initial conditions corresponding to the image rays, (iii) the fact that our finite-difference scheme contains small error terms which damp the high harmonics; truncation of the Chebyshev series, and (iv) the need to compute the solution only for a short interval of time. We test our numerical schemes on a collection of analytic examples and demonstrate a dramatic improvement in accuracy in the estimation of the sound speed inside the Earth in comparison with the conventional Dix inversion. Our test on the Marmousi example confirms the effectiveness of the proposed approach.

1. Introduction

In the present work we derive and study a non-linear elliptic PDE for seismic velocity estimation from time migration. The physical setting allows us to pose only a Cauchy problem and this is ill-posed. Nonetheless, because this PDE provides an inexpensive way to estimate the sound speed inside the Earth, an attempt to provide some sort of solution is worthwhile. We begin with a short overview.

Seismic data are the records of the sound wave amplitudes $P(S,G,t)$ where S is the source position, G is the receiver position, and t is the time. Seismic reflection imaging can be viewed as a procedure of obtaining the amplitude at the subsurface point (x,y,z) from the data points (S,G,t), where $(x,y,z) = R$ is the reflection point of the ray path from the source S to the receiver G (see Fig. 1).

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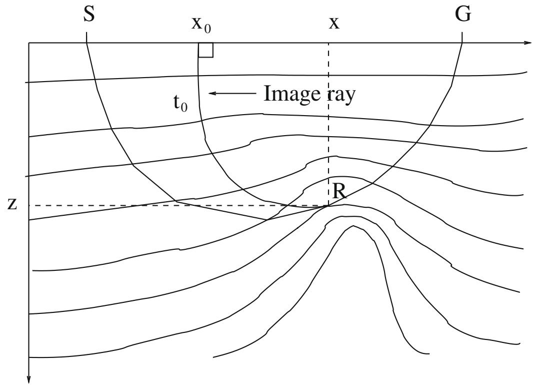

Fig. 1. The raypath between the source S, the reflection point R and the receiver G; the image ray from the reflection point R and the time and depth coordinates of the point R.

To obtain an accurate image at the reflection point $R = (x, y, z)$, one needs to sum up all of the recorded responses from the point R in the data domain with certain weights. Such a weighted summation of the amplitudes in the data domain is the essence of the so-called *Kirchhoff prestack depth migration* [30]. In order to extract the responses from every single reflection point from the set of the recorded data, one needs to know the traveltimes from every source S to every reflection point $R = (x, y, z)$ and from every reflection point R to every receiver G. For computing such traveltimes, one needs to have a *velocity model in depth* $v(x, y, z)$, i.e, the speed of the propagation of the seismic waves inside the earth. We call such a model *seismic velocity*. In the case of an isotropic seismic velocity, one can solve the eikonal equation

$$|\nabla T(x, y, z)|^2 = \frac{1}{v^2(x, y, z)} \tag{1}$$

to find the desired traveltimes.

The major problem of seismic imaging is that such a velocity model is hard to build. A number of powerful automatic velocity estimation methods have been proposed. This includes reflection tomography [29], stereotomography [14], migration velocity analysis [26,27], and differential semblance optimization [25]. However, these methods typically involve considerable computational expense and rely on a good initial approximation. Numerical studies of the well-known Marmousi data [28] demonstrate that, in the absence of a good initial guess, none of the modern approaches are fully reliable. The approach which this paper is concerned with is computationally cheap and requires no initial guess. It can provide an initial guess for the approaches listed above.

In [3] we formulated an inverse problem of finding the seismic velocities from the so-called “Dix velocities”, and showed that it is ill-posed in the sense that small perturbations in the Dix velocity may lead to big changes in the seismic velocity. Nevertheless, in that paper we also attempted a regularized reconstruction and developed two numerical approaches to solve the problem. Since the estimated seismic velocity was used in the depth migration, only for the computation of the traveltimes, and was not used for the delineation of the subsurface reflectors, smoothing of the velocity model did not lead to significant errors.

The key problem in these approaches hinged on the estimation of the second derivatives of the unknown velocity. We used a least squares polynomial approximation to regularize the solution. However, choosing the degree of the least squares polynomials was sensitive. If the degree was too high, oscillations developed; if it was too low, the solution was inexact.

In this work, we develop novel inversion methods which involve neither the least squares polynomial approximation nor ray tracing. Our results include the following:

- In the theoretical part, we derive a partial differential equation for Q which is the geometrical spreading of image rays [11], and involves only the Dix velocity and its derivatives with respect to the starting surface points and time. This reformulated PDE reveals the nature of the instabilities in the problem in hand. The PDE is elliptic, and the physical setting allows us to pose only a Cauchy problem, which is known to be ill-posed. Furthermore, the fact that the PDE involves not only the Dix velocity itself but also its first and second derivatives leads to high sensitivity to the input data. This makes the ill-posedness analysis given in [3] unsurprising: a small perturbation of the Dix velocity can produce a significant corresponding change in its second derivative, and can lead to a considerable change in the seismic velocity.
- Despite the fact that problem is ill-posed, we show that we are still able to find a way to compute the solution:
  - First, we develop a finite difference time-marching numerical scheme and compute a solution on the required interval of time. Our numerical scheme is motivated by the Lax–Friedrichs [15] method for hyperbolic conservation laws as a building block.
  - Second, we adjust a spectral Chebyshev method for the problem in-hand. We truncate the Chebyshev series to cut off the growing high harmonics in this case.