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POD/DEIM nonlinear model order reduction of an ADI implicit shallow water equations model

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ABSTRACT

In the present paper we consider a 2-D shallow-water equations (SWE) model on a β-plane solved using an alternating direction fully implicit (ADI) finite-difference scheme on a rectangular domain. The scheme was shown to be unconditionally stable for the linearized equations.

The discretization yields a number of nonlinear systems of algebraic equations. We then use a proper orthogonal decomposition (POD) to reduce the dimension of the SWE model. Due to the model nonlinearities, the computational complexity of the reduced model still depends on the number of variables of the full shallow – water equations model. By employing the discrete empirical interpolation method (DEIM) we reduce the computational complexity of the reduced order model due to its depending on the nonlinear full dimension model and regain the full model reduction expected from the POD model.

To emphasize the CPU gain in performance due to use of POD/DEIM, we also propose testing an explicit Euler finite difference scheme (EE) as an alternative to the ADI implicit scheme for solving the swallow water equations model.

We then proceed to assess the efficiency of POD/DEIM as a function of number of spatial discretization points, time steps, and POD basis functions. As was expected, our numerical experiments showed that the CPU time performances of POD/DEIM schemes are proportional to the number of mesh points. Once the number of spatial discretization points exceeded 10000 and for 90 DEIM interpolation points, the CPU time decreased by a factor of 10 in case of POD/DEIM implicit SWE scheme and by a factor of 15 for the POD/DEIM explicit SWE scheme in comparison with the corresponding POD SWE schemes. Moreover, our numerical tests revealed that if the number of points selected by DEIM algorithm reached 50, the approximation errors due to POD/DEIM and POD reduced systems have the same orders of magnitude, thus supporting the theoretical results existing in the literature.

1. Introduction

The shallow water equations are the simplest form of the equations of motion that can be used to describe the horizontal (motion) structure of the atmosphere. They describe the evolution of an incompressible and inviscid fluid in response to gravitational and rotational accelerations and their solutions represent East West propagating Rossby waves and inertia – gravity waves.

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To avoid the limitations imposed by the Courant Friedrichs–Lewy (CFL) stability conditions restricting the time steps in explicit finite difference approximations, implicit scheme must be considered. We propose here the alternating direction implicit (ADI) method introduced by Gustafsson [1]. Linear and nonlinear versions of ADI scheme may be found in studies proposed by Fairweather and Navon [2] and Navon and De Villiers [3]. Kreiss and Widlund [4] established the convergence of alternating direction implicit methods for elliptic problems. Such methods reduce multidimensional problem to systems of one dimensional problems (Douglas and Gunn [5], Yanenko [6] and Marchuck [7]).

The nonlinear algebraic systems corresponding to the discrete model were solved using the quasi-Newton method proposed in Gustafsson [1]. This quasi-Newton method performs an LU decomposition done every Mth time step, where M is a fixed integer. Since back substitution is a fast operation the scheme will be efficient as long as the number of iterations is small.

The major issue in large scale complex modelling is that of reducing the computational cost while preserving numerical accuracy. Among the model reduction techniques, the proper orthogonal decomposition (POD) method provides an efficient means of deriving the reduced basis for high-dimensional nonlinear flow systems. The POD method has been widely and successfully applied to signal analysis and pattern recognition as Karhunen–Loève, statistics as principal component analysis (PCA), geophysical fluid dynamics and meteorology as empirical orthogonal functions (EOF) etc. The POD method was applied also to SWE model and we mention here the work of Cao et al. [8], Vermeulen and Heemink [9], Daescu and Navon [10] and Altaf et al. [11].

In this paper we reduced the dimension of the SWE model by employing the POD method. However due to the nonlinearities of the implicit SWE model the computational complexity of the reduced shallow water equations model still depends on the number of variables of the full shallow – water equations model. To mitigate this problem, we apply the discrete empirical interpolation method (DEIM) to address the reduction of the nonlinear components and thus reduce the computational complexity by implementing the POD/DEIM method.

DEIM is a discrete variant of the empirical interpolation method (EIM) proposed by Barrault et al. [12] for constructing an approximation of a non-affine parameterized function, which was proposed in the context of reduced-basis discretization of nonlinear partial differential equations. The application was suggested and analysed by Chaturantabut and Sorensen [13–16].

The paper is organized as follows. In Section 2 we introduce the Gustafsson ADI fully implicit method applied to the shallow water equations model and briefly describe its algorithmic components since they are already available in archived literature. In Section 3 we describe in some detail the snapshot POD procedure and its implementation to the ADI method for the SWE model. Section 4 addresses the snapshot POD combined with DEIM methodology and provides the detailed algorithmic description of the DEIM implementation. In Section 5 we present the numerical experiments related to the POD/DEIM procedure for both explicit and implicit schemes applied to the SWE models.

The POD/DEIM procedure amounts to replace orthogonal projection with an interpolation projection of the nonlinear terms that requires the evaluation of only a few selected components of the nonlinear terms.

We evaluate the efficiency of DEIM as a function of number of spatial discretization points, time steps and basis functions for this quadratically nonlinear problem and additional studies about the conservation of the integral invariants of the SWE, root mean square errors (RMSEs) and correlation coefficients between full model, POD and POD/DEIM systems were performed.

2. Brief description of the Gustafsson ADI method

In meteorological and oceanographic problems, one is often not interested in small time steps because the discretization error in time is small compared to the discretization error in space. The fully implicit scheme considered in this paper is first order in both time and space and it is stable for large CFL condition numbers. It was proved by Gustafsson [1] that the method is unconditionally stable for the linearized version of the SWE model.

Here we shortly describe the Gustafsson shallow water alternating direction implicit method Gustafsson [1], Fairweather and Navon [2], Navon and de Villiers [3]. We are solving the SWE model using the β-plane approximation on a rectangular domain.

$$\frac{\partial w}{\partial t} = A(w)\frac{\partial w}{\partial x} + B(w)\frac{\partial w}{\partial y} + C(y)w, \tag{1}$$

$$0 \leqslant x \leqslant L, \quad 0 \leqslant y \leqslant D, \quad t \in (0, t_f],$$

where $w = (u, v, \phi)^T$ is a vector function, u, v are the velocity components in the x and y directions, respectively, h is the depth of the fluid, g is the acceleration due to gravity and $\phi = 2\sqrt{gh}$.

The matrices A, B and C are expressed

$$A = -\begin{pmatrix} u & 0 & \phi/2 \\ 0 & u & 0 \\ \phi/2 & 0 & u \end{pmatrix}, \quad B = -\begin{pmatrix} v & 0 & 0 \\ 0 & v & \phi/2 \\ 0 & \phi/2 & v \end{pmatrix}, \quad C = \begin{pmatrix} 0 & f & 0 \\ -f & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$