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Lattice Boltzmann magnetohydrodynamics with current-dependent resistivity

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ABSTRACT

Lattice Boltzmann magnetohydrodynamics is extended to allow the resistivity to be a prescribed function of the local current density. Current-dependent resistivities are used to model the so-called anomalous resistivity caused by unresolved small-scale processes, such as current-driven plasma microturbulence, that are excluded by the magnetohydrodynamics approximation. These models closely resemble the Smagorinsky eddy viscosity model used in large eddy simulations of hydrodynamic turbulence. Lattice Boltzmann implementations of the Smagorinsky model adjust the collision time in proportion to the local rate of strain, as obtained from the non-equilibrium parts of the hydrodynamic distribution functions. This works successfully even with a single relaxation time collision operator. However, the existing lattice Boltzmann magnetohydrodynamic implementation contains a spurious term in the evolution equation for the magnetic field that violates the divergence-free condition when the relaxation time varies in space. A correct implementation requires a matrix collision operator for the magnetic distribution functions. The relaxation time imposed on the antisymmetric component of the electric field tensor is calculated locally from the non-equilibrium part of the magnetic distribution functions, which determine the current, while the symmetric component remains subject to a uniform relaxation time to suppress the spurious term. The resulting numerical solutions are shown to converge to independent spectral solutions of the magnetohydrodynamic equations, and to preserve the divergence-free condition up to floating point round-off error.

1. Introduction

The lattice Boltzmann method has been widely used for simulations of hydrodynamic [1,2] and magnetohydrodynamic (MHD) turbulence, notably a large-scale 1800^3 simulation of isotropic magnetohydrodynamic turbulence [3]. Recent simulations show very close agreement between low order statistics computed from lattice Boltzmann and spectral simulations of homogeneous, isotropic turbulence [2]. Besides direct numerical simulation (DNS) of the Navier–Stokes and MHD equations, the lattice Boltzmann approach has been used to perform large eddy simulations (LES) of hydrodynamic turbulence using the Smagorinsky eddy viscosity model [4–8]. The Smagorinsky model describes the effects of unresolved small-scale turbulence through an extra eddy viscosity

$$\nu_S = \ell^2 ||\mathbf{S}|| \tag{1}$$

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proportional to the norm $||\mathbf{S}|| = (\mathbf{S}:\mathbf{S})^{1/2}$ of the local fluid strain rate tensor $\mathbf{S} = \nabla\mathbf{u} + (\nabla\mathbf{u})^{\mathsf{T}}$. The mixing length $\ell = C_S\Delta x$ is related to the grid spacing Δx through the dimensionless Smagorinsky constant C_S, typically taken to be around 0.1. The coefficient C_S may also be calculated dynamically by applying different levels of filtering to the finest resolved velocity field [9–12]. If one instead treats the mixing length ℓ as fixed, the Smagorinsky model falls within the class of generalised Newtonian fluids whose viscosities are functions of the magnitude of the local strain rate [13]. Lattice Boltzmann simulations of these fluids have been widely applied to geological and physiological flows [14–18].

The lattice Boltzmann approach [19,20] to fluid simulation represents the hydrodynamic variables, such as density ρ, velocity $\mathbf{u}$, and momentum flux $\mathbf{\Pi}$, as moments of a set of distribution functions f_i,

$$\rho = \sum_{i=0}^{N} f_i, \quad \rho\mathbf{u} = \sum_{i=0}^{N} \boldsymbol{\xi}_i f_i, \quad \mathbf{\Pi} = \sum_{i=0}^{N} \boldsymbol{\xi}_i\boldsymbol{\xi}_i f_i. \tag{2}$$

The f_i evolve according to a discrete Boltzmann equation of the form

$$\partial_t f_i + \boldsymbol{\xi}_i \cdot \nabla f_i = -\sum_{j=0}^{N} \Omega_{ij}(f_j - f_j^{(0)}), \tag{3}$$

where the constant vectors $\boldsymbol{\xi}_i$ are discrete velocities associated with each f_i. The left hand side of this equation is a linear, constant coefficient differential operator that may be readily discretised by integration along its characteristics [21–24]. All nonlinearity is confined to the collision term on the right hand side, which may be implemented locally grid point by grid point. These properties render the lattice Boltzmann approach particularly amenable to efficient implementation on massively parallel computers and graphics processing units (GPUs).

The equilibrium distributions $f_j^{(0)}(\rho, \mathbf{u})$ and collision matrix Ω_{ij} are chosen so that slowly varying solutions of the moment hierarchy

$$\partial_t\rho + \nabla\cdot(\rho\mathbf{u}) = 0, \quad \partial_t(\rho\mathbf{u}) + \nabla\cdot\mathbf{\Pi} = 0, \quad \partial_t\mathbf{\Pi} + \nabla\cdot\left(\sum_{i=0}^{N}\boldsymbol{\xi}_i\boldsymbol{\xi}_i\boldsymbol{\xi}_i f_i\right) = -\frac{1}{\tau}(\mathbf{\Pi} - \mathbf{\Pi}^{(0)}) \tag{4}$$

obtained from (3) satisfy the isothermal compressible Navier–Stokes equations on timescales much longer than the timescale τ associated with collisions. The momentum flux in these slowly varying solutions, as obtained using the multiple-scales Chapman–Enskog expansion [25,26] is

$$\mathbf{\Pi} = \mathbf{\Pi}^{(0)} - \tau\rho c_s^2\mathbf{S} + O(\tau^2), \tag{5}$$

where $\mathbf{\Pi}^{(0)} = \rho c_s^2\mathsf{I} + \rho\mathbf{u}\mathbf{u}$ is the Euler momentum flux calculated from the $f_i^{(0)}$. The sound speed c_s is constant in an isothermal fluid, and I is the identity tensor. The $O(\tau)$ correction to $\mathbf{\Pi}^{(0)}$ gives a Newtonian viscous stress with dynamic viscosity $\mu = \tau\rho c_s^2$. It is thus possible to calculate

$$\mathbf{S} = -\frac{1}{\tau\rho c_s^2}\sum_{i=0}^{N}\boldsymbol{\xi}_i\boldsymbol{\xi}_i(f_i - f_i^{(0)}), \tag{6}$$

locally from the nonequilibrium parts of the distribution functions at each grid point. This offers an attractive alternative to finite difference approximations for differentiating the velocity field. The Smagorinsky model is thus readily incorporated in the lattice Boltzmann approach by making the collision time τ a function of the local strain rate [5–7]. However, a second-order accurate implementation is complicated by the fact that the lattice Boltzmann numerical algorithm evolves a transformed set of distribution functions $\bar{f}_i$, as described in Section 6, rather than the f_i that appear in (6). This results in a Hénon correction [27] of τ to $\tau + \Delta t/2$ in the denominator of (6) for a lattice Boltzmann algorithm with timestep Δt. It is thus necessary to solve a nonlinear algebraic equation at each grid point to determine $||\mathbf{S}||$ [8,18].

A Smagorinsky-like eddy resistivity model has been employed in large eddy simulations of magnetohydrodynamic turbulence [28–30]. More generally, we consider an Ohm's law in which the resistivity η is a function of the local current density $\mathbf{J}$,

$$\mathbf{E} + \mathbf{u}\times\mathbf{B} = \eta(|\mathbf{J}|)\mathbf{J}, \tag{7}$$

where $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic fields. It is convenient to use $\mu_0 = 1$ units in which $\mathbf{J} = \nabla\times\mathbf{B}$. Ohm's law is the MHD analogue of the Navier–Stokes constitutive relation between stress and strain rate. Substituting the expression for $\mathbf{E}$ given by (7) into Faraday's law $\partial_t\mathbf{B} + \nabla\times\mathbf{E} = 0$ gives an evolution equation for the magnetic field,

$$\partial_t\mathbf{B} = \nabla\times(\mathbf{u}\times\mathbf{B} - \eta\nabla\times\mathbf{B}), \tag{8}$$

which may be rewritten as

$$\partial_t\mathbf{B} = \nabla\times(\mathbf{u}\times\mathbf{B}) - \eta\nabla\times\mathbf{J} - \nabla\eta\times\mathbf{J}. \tag{9}$$

The last term involving $\nabla\eta\times\mathbf{J}$ is only present when η is spatially varying.