Journal of Computational Physics 228 (2009) 4945–4960

Contents lists available at ScienceDirect

Journal of Computational Physics

journal homepage: www.elsevier.com/locate/jcp

The finite volume local evolution Galerkin method for solving the hyperbolic conservation laws

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ARTICLE INFO

Article history:
Received 14 May 2008
Received in revised form 31 March 2009
Accepted 4 April 2009
Available online 11 April 2009

MSC:
65M06
76N15
35L65

Keywords:
Multi-dimensional upwind scheme
Finite volume scheme
Euler equations
The finite volume local evolution Galerkin method

ABSTRACT

This paper presents a finite volume local evolution Galerkin (FVLEG) scheme for solving the hyperbolic conservation laws. The FVLEG scheme is the simplification of the finite volume evolution Galerkin method (FVEG). In FVEG, a necessary step is to compute the dependent variables at cell interfaces at $t_n + \tau$ $(0 < \tau \leqslant \Delta t)$. The FVLEG scheme is constructed by taking $\tau \to 0$ in the evolution operators of FVEG. The FVLEG scheme greatly simplifies the evaluation of the numerical fluxes. It is also well suited with the semi-discrete finite volume method, making the flux evaluation being decoupled with the reconstruction procedure while maintaining the genuine multi-dimensional nature of the FVEG methods. The derivation of the FVLEG scheme is presented in detail. The performance of the proposed scheme is studied by solving several test cases. It is shown that FVLEG scheme can obtain very satisfactory numerical results in terms of accuracy and resolution.

1. Introduction

The numerical solutions of systems of hyperbolic conservation laws have been dominated by Riemann-solver-based schemes since the work of Godunov [4], Van Leer [22], Harten–Lax [5], Osher and Solomon [15] and Roe [17]. This approach, known as flux-difference splitting (FDS), has the desirable property of accurately resolving shock waves as well as contact discontinuities. When extending the flux-difference schemes to multi-dimensional problems, the so-called grid aligned finite volume approach or dimensional splitting method is adopted traditionally using one-dimensional Riemann solvers. However, for multi-dimensional problem, there is in general no longer a finite number of directions of information propagation. Roe [18] has pointed out that the approach based on one-dimensional Riemann solvers may lead to a misinterpretation of the local wave structure of the solution. In fact, it turned out that in certain cases, e.g. when there are strong shocks or waves are propagating in directions that are oblique with respect to the mesh, this approach leads to structural deficiencies and large errors in the solutions [8,16].

To overcome the drawbacks of existing methods based on dimensional splitting or the "grid-aligned" approaches, there have been considerable efforts to develop so-called "genuinely multi-dimensional schemes" for solving hyperbolic conservation laws in recent years [2,3,6,14]. While we are not in the position to give a detailed review of these schemes, we would

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0021-9991/$ - see front matter
doi:10.1016/j.jcp.2009.04.001

like to draw the readers' attention to the genuinely multi-dimensional finite volume evolution Galerkin (FVEG) method [11] which is also the starting point of the present paper.

The finite volume evolution Galerkin (FVEG) schemes can be considered as a generalization of the original idea of Godunov to multi-dimensional hyperbolic conservation laws within the framework of the finite volume approach. To construct genuinely multi-dimensional schemes, the exact integral equations from a general theory of bicharacteristics for linear (or linearized) hyperbolic systems were derived. These integral equations were further approximated by approximate evolution operators in such a way that all of the infinitely many directions of propagation of bicharacteristics were explicitly taken into account. These approximate evolution operators were then used to compute the interfacial dependent variables for the evaluation of the numerical fluxes.

The FVEG schemes have been studied extensively from theoretical as well as numerical point of view and applied to various applications [9,10,12,21]. It is shown that the FVEG schemes yield better accuracy and resolution than some well known finite difference and finite volume schemes. However, theses schemes are more complicated in implementation than traditional finite volume schemes. For two-dimensional FVEG schemes, the numerical fluxes across cell interfaces are computed preferably by the Simpson rule. The use of the Simpson rule takes the multi-dimensional effects at cell vertices into account and is beneficial to the monotonicity of the scheme [9]. Using this approach, the complication comes mainly from the evaluation of the values of the dependent variables at $t_n + \tau$ $(0 < \tau \leqslant \Delta t)$ at the midpoint and two corner points of a cell interface. In practice, these interfacial values of the dependent variables are evaluated by certain approximate evolution operators which involve the integrals around the Mach cones. These integrals can be computed exactly as well as numerically. However, for slant Mach cones associated with the nonlinear hyperbolic systems (e.g. the Euler equations), the exact evaluation of the integrals leads to very lengthy and tedious computations, especially when reconstructions with higher order polynomials are adopted in the finite volume schemes. Using numerical integrations may simplify the computation; however, it also leads to an increase of computational cost and/or a decrease of accuracy especially when the reconstruction functions are discontinuous at cell interfaces.

In the present paper, a finite volume local evolution Galerkin (FVLEG) method is proposed. The FVLEG method is a combination of the FVEG method and the semi-discrete finite volume scheme, in which a necessary step is to let $\tau \to 0$ in the evolution operators of FVEG. It is shown that the FVLEG approach greatly simplifies the evaluation of numerical fluxes and also makes it straightforward to apply the FVLEG scheme on general shaped control volumes. Furthermore, because of the semi-discrete nature of the present method, the flux evaluation is decoupled with the reconstruction procedure and time integration is independent of the spatial discretiztion. These properties are important in constructing both temporally and spatially higher order schemes. The performance of the proposed scheme is studied by solving several test cases. It is shown that FVLEG scheme can obtain very good numerical results in terms of both accuracy and resolution.

2. The finite volume schemes

2.1. The governing equations

Although the FVEG schemes can be applied to general hyperbolic conservation laws, we consider here the two-dimensional Euler equations describing the compressible inviscid flows without a loss of the generality. In conservation form the Euler equations are

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} + \frac{\partial \mathbf{G}}{\partial y} = 0, \tag{1}$$

where $\mathbf{U}$ is the vector of the conserved variables given as $\mathbf{U} = [\rho, \rho u, \rho v, \rho E]^{\mathrm{T}}$. The detailed formulations of the flux terms are well known and are omitted here for brevity.

2.2. The finite volume scheme

We consider some two-dimensional domain in x–y space and assume that it is discretized into structured quadrilateral control volumes. Examples of typical control cells are shown in Fig. 1. Finite volume schemes for Eq. (1) are obtained by considering the control volume balance equations

$$\frac{\partial}{\partial t}\int\int_{\Omega_{ij}} \mathbf{U}_{ij}\,dx\,dy + \oint_{\partial\Omega_{ij}} \mathbf{H}\cdot\mathbf{n}\,dl = 0, \tag{2}$$

where Ω_{ij} is the control volume, $\partial\Omega_{ij}$ is the boundary of Ω_{ij}, $\mathbf{H} = \mathbf{F}\mathbf{i} + \mathbf{G}\mathbf{j}$ is the tensor of the fluxes. $\mathbf{n} = n_x\mathbf{i} + n_y\mathbf{j}$ is the outward unit vector normal to the surface $\partial\Omega_{ij}$. On a quadrilateral control volume with its faces denoted by $I_k = I_{i+\alpha(k),j+\beta(k)}$ $(k = 1,\ldots,4)$, where $\alpha(k) = \frac{1}{2}\sin\left(\frac{(k-2)\pi}{2}\right), \beta(k) = \frac{1}{2}\cos\left(\frac{k\pi}{2}\right)$, the finite volume balance equations can be written as

$$\frac{\partial \overline{\mathbf{U}}_{ij}}{\partial t} = -\frac{1}{\overline{\Omega}_{ij}}\sum_{k=1}^{4}\int_{I_k}\mathbf{H}\cdot\mathbf{n}\,dl, \tag{3}$$