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# Compact third-order limiter functions for finite volume methods

Miroslav Čada *, Manuel Torrilhon

*Seminar for Applied Mathematics, ETH Zentrum, CH-8092 Zurich, Switzerland*

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ABSTRACT

We consider finite volume methods for the numerical solution of conservation laws. In order to achieve high-order accurate numerical approximation to non-linear smooth functions, we introduce a new class of limiter functions for the spatial reconstruction of hyperbolic equations. We therefore employ and generalize the idea of double-logarithmic reconstruction of Artebrant and Schroll [R. Artebrant, H.J. Schroll, Limiter-free third order logarithmic reconstruction, SIAM J. Sci. Comput. 28 (2006) 359-381].

The result is a class of efficient third-order schemes with a compact three-point stencil. The interface values between two neighboring cells are obtained by a single limiter function. The limiter belongs to a family of functions, which are based upon a non-polynomial and non-linear reconstruction function. The new methods handle discontinuities as well as local extrema within the standard semi-discrete TVD-MUSCL framework using only a local three-point stencil and an explicit TVD Runge–Kutta time-marching scheme. The shape-preserving properties of the reconstruction are significantly improved, resulting in sharp, accurate and symmetric shock capturing. Smearing, clipping and squaring effects of classical second-order limiters are completely avoided.

Computational efficiency is enhanced due to large allowable Courant numbers ($\mathrm{CFL} \lesssim 1.6$), as indicated by the von Neumann stability analysis. Numerical experiments for a variety of hyperbolic partial differential equations, such as Euler equations and ideal magneto-hydrodynamic equations, confirm a significant improvement of shock resolution, high accuracy for smooth functions and computational efficiency.



## 1. Introduction

In the present paper we will derive a new third-order method for the computation of numerical fluxes within the framework of finite volume (FV) schemes for conservation laws. FV methods, in general, are derived from the integral formulation of conservation laws for a spatial domain $\Omega$ and a quantity $\boldsymbol{u}$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega}\boldsymbol{u}(x,t)\,\mathrm{d}x = -\int_{\partial\Omega}\boldsymbol{f}(\boldsymbol{u}(x,t))\cdot\boldsymbol{n}\,\mathrm{d}S. \tag{1.1}$$

The volume of $\boldsymbol{u}$ can only change in time by the dynamics of the flux $\boldsymbol{f}(\boldsymbol{u}(x,t))$ across the boundary $\partial\Omega$.

Solutions to these equations have typically smooth structures interspersed with discontinuities. An accurate prediction of such interactions is of importance in many computational fluid dynamic (CFD) applications, such as aircraft design, stellar formation, and weather simulations, to name only a few. The main task is to develop algorithms that ensure the conservation property Eq. (1.1), that are highly accurate for smooth regions in both time and space, and that have sharp transition where large gradients or discontinuities appear.

* Corresponding author.
*E-mail addresses:* cada@math.ethz.ch (M. Čada), matorril@math.ethz.ch (M. Torrilhon).

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Numerical methods for conservation laws can in general be divided into two approaches: a one-step approach, where time and space discretization is coupled via the Cauchy–Kowalewski procedure (see e.g. [15,27]) and semi-discrete schemes, where time and space integration are separated. The latter has been a very successful approach towards the dual objective, i.e., high accuracy and non-oscillatory transitions. It uses strong stability preserving Runge–Kutta time-marching schemes, combined with polynomial reconstruction of the interface value $\boldsymbol{u}(x,t)|_{\partial\Omega}$ of the flux $\boldsymbol{f}(\boldsymbol{u}(x,t)|_{\partial\Omega})$ from a given cell average.

One of the first high-order semi-discrete algorithm was van Leer's TVD-MUSCL (Monotonic Upstream-centered Scheme for Conservation Laws) scheme [19]. A piecewise-linear spatial approximation was combined with a second-order time integration. To avoid spurious oscillations total variation diminishing (TVD) limiters were used. TVD limiters are bounded non-linear functions obeying Harten's TVD conditions [8]. These limiters ensure that any reconstructed values at any time do not lie outside the range of the initial data, thus precluding accurate resolution of local extrema. To avoid this drawback, Harten et al. [9] later introduced the concept of essentially non-oscillatory (ENO) schemes. ENO methods use instead of a non-linear limiter an optimal stencil search procedure, which chooses the locally "smoothest" reconstruction, thereby avoiding interpolation across discontinuities. Due to the adaptive stencil, the finally approximated interface value does not consist of all available data. In the weighted ENO (WENO) scheme by Liu et al. [17] the interface values are computed as convex combinations of all candidate stencils, hence making better use of the available data. In contrast to TVD schemes both ENO and WENO methods introduce spurious yet bounded variation, which eventually decreases as the mesh is refined.

At the same time the idea of limiting has been further developed as extensions of the TVD-MUSCL scheme. A third-order method was developed by Woodward and Colella [36]. They used a four-point centered stencil to recover the interface values, which afterwards were limited in the vicinity of a discontinuity. An even higher-order extension was introduced by Suresh and Huynh [26] employing a five-point stencil reconstruction with a limiting procedure, which preserves monotonicity and accuracy for smooth extrema, easing the TVD constrain. To apply the costly limiting routine efficiently, they preprocess the data to localize discontinuities. Waterson and Deconinck [34] have recently presented a unified classification of limiters based on three-points. A successful limiter in this family is the Koren limiter [13] (see also [2]). All analyzed limiter functions were derived from polynomial reconstruction functions and degenerated to first-order for smooth extrema.

In contrast to the method based upon a polynomial approximation of the interface values, Artebrant and Schroll [5] have recently developed a new non-polynomial reconstruction function. Originally motivated by the work of Marquina [18], who used a hyperbola as reconstruction function, they recover the interface values with a local double logarithmic (LDLR) Ansatz function. Due to the logarithmic nature it is able to handle discontinuities without spurious oscillations, yet introducing local bounded variation. In contrast to Marquina's hyperbolic reconstruction LDLR can recover local extrema without loss of accuracy. It is local, in the sense that it uses only a three-point stencil and it does not use classical limiters to avoid oscillations. Numerical experiments in [5] indicate its superior performance compared to other third-order methods and reconstruction techniques.

Inspired by the results of Artebrant and Schroll [5] we have followed a different path. The local and symmetric nature of LDLR allowed a simpler, yet more efficient formulation of Artebrant and Schroll's scheme, thus resembling very much van Leer's three-point TVD-MUSCL algorithm using only a single limiter function. Furthermore this interpretation not only simplified the original reconstruction, but also improved its shock capturing qualities and reconstruction efficiency without the application of a logarithmic function at all. This eventually led to a whole family of limiter functions, which recover smooth extrema with third-order accuracy. We want to emphasize that the paradigm of this paper is to rigorously restrict the limiting procedure to a three-point stencil. Consequently we have to cope with the problem of recovering discretized smooth structures with few data points. Yet, we remain in the well-understood context of the efficient classical TVD-MUSCL scheme. Taking into account that the proposed limiters are simple $(\max -, \min -)$ functions, which do not expand the computational stencil, they become attractive candidates to be easily incorporated into already existing TVD codes.

To emphasize the close relation of the proposed third-order limiting method with van Leer's TVD-MUSCL scheme, we first revisit the second-order one in Section 2. In Section 3, the main properties of the new limiter functions – the key features of our scheme – its connection and the differences to LDLR and other schemes will be explained. Furthermore the difficulty of recovering local extrema with a three-point stencil without loss of accuracy will be discussed and solved. In order to reach a overall third-order accuracy we use the explicit three-stage TVD Runge–Kutta time integration of Gottlieb and Shu [7]. In Section 4 we investigate the characteristic properties of the new methods in the frequency domain, in terms of a linear von Neumann stability analysis. Convergence studies and various numerical experiments, including one- and two-dimensional systems appear in Section 5. Finally conclusions are presented in Section 6.

## 2. Spatial reconstruction: standard TVD-MUSCL methods

For simplicity and in order to settle our notation we consider the numerical approximations to the one-dimensional scalar initial value problem

$$\begin{aligned} &u_t = -f(u)_x, \\ &u(x,t=0) = u_0(x), \end{aligned} \tag{2.1}$$

where $u_0$ is either a piecewise smooth function with compact support or a periodic function. Note that the flux in Eqs. (1.1) and (2.1) appears on the right-hand side to emphasize the semi-discrete formulation. We cover the uniform computational