Journal of Computational Physics 228 (2009) 4273–4290

Contents lists available at ScienceDirect

Journal of Computational Physics

journal homepage: www.elsevier.com/locate/jcp

High order numerical methods to two dimensional delta function integrals in level set methods ☆

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ARTICLE INFO

Article history:
Received 4 May 2008
Received in revised form 9 January 2009
Accepted 8 March 2009
Available online 17 March 2009

AMS subject classification:
65Jxx

Keywords:
Delta function integral
High order numerical method
Error estimate
Level set method

ABSTRACT

In this paper we design and analyze a class of high order numerical methods to delta function integrals appearing in level set methods in two dimensional case. The methods comprise approximating the mesh cell restrictions of the delta function integral. In each mesh cell the two dimensional delta function integral can be rewritten as a one dimensional ordinary integral with the smooth integrand being a one dimensional delta function integral, and thus is approximated by applying standard one dimensional high order numerical quadratures and high order numerical methods to one dimensional delta function integrals proposed in [X. Wen, High order numerical methods to a type of delta function integrals, J. Comput. Phys. 226 (2007) 1952–1967]. We establish error estimates for the method which show that the method can achieve any desired accuracy by assigning the corresponding accuracy to the sub-algorithms and has better accuracy under an assumption on the zero level set of the level set function which holds generally. Numerical examples are presented showing that the second to fourth order methods implemented in this paper achieve or exceed the expected accuracy and demonstrating the advantage of using our high order numerical methods.

1. Introduction

We study in this paper a class of high order numerical methods to the following type of delta function integrals

$$\int_{\mathbb{R}^n} f(\mathbf{x})\|\nabla u(\mathbf{x})\|\delta(u(\mathbf{x}))d\mathbf{x}, \tag{1.1}$$

where $f(\mathbf{x})$ is a weight function, $u(\mathbf{x})$ is a level set function whose zero points define a manifold Γ of codimension one. In this paper we consider the two dimensional case $n = 2$. In this case Γ is a one dimensional curve in the two dimensional space. The functions $f(\mathbf{x}), u(\mathbf{x})$ are assumed to have sufficient smoothness and their values are only provided at grid points of a regular mesh. Numerical evaluations of delta function integrals (1.1) in two and three dimensions in the above context appear in many applications of level set methods, see for example [2,11,26].

One approach widely studied in the literature to approximate (1.1) is the numerical quadrature approach. Assume the values of $f(\mathbf{x}), u(\mathbf{x})$ are given at grid points of the following uniform mesh on $\mathbb{R}^2$

☆ Research supported in part by the Knowledge Innovation Project of the Chinese Academy of Sciences Grants K5501312S1, K5502212F1, K7290312G7 and K7502712F7, and NSFC Grant 10601062.

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doi:10.1016/j.jcp.2009.03.004

$$\{\mathbf{x_j}\}_{\mathbf{j}\in\mathbb{Z}^2},\quad \mathbf{x_j} = \left(x^{(1)}_{j_1}, x^{(2)}_{j_2}\right),$$
$$x^{(k)}_{j_k} = x^{(k)}_0 + j_k h,\quad j_k \in \mathbb{Z}, k = 1, 2. \tag{1.2}$$

Then the numerical quadrature methods to delta function integrals (1.1) in two dimension can be generally written in the form

$$h^2 \sum_{\mathbf{j}\in\mathbb{Z}^2} f(\mathbf{x_j}) \|\nabla_D u(\mathbf{x_j})\| \tilde{\delta}(\mathbf{x_j}; u), \tag{1.3}$$

where $\nabla_D u(\mathbf{x_j})$ is the difference approximation to $\nabla u(\mathbf{x_j})$, $\tilde{\delta}(\mathbf{x_j}; u)$ is an approximate delta function which can be defined by grid point values of u. Utilizing approximate delta functions is related to the regularization technique of the Dirac delta function which have various applications [4–7,10,12–14,16–18,21,22]. The efficiency of the numerical quadratures (1.3) depends on the choice of appropriate approximate delta functions. In the work [18] Tornberg and Engquist showed that a common technique to construct the approximate delta function in (1.3) suffers from $O(1)$ errors. After that different approximate delta functions have been designed in the literature which yield first or second order numerical quadratures (1.3). In [3] Engquist et al. proposed a first order accurate approximate delta function based on one dimensional discrete delta functions and a variable support size formula. They also developed in [3] a second order accurate approximate delta function based on approximations to product formula for multidimensional delta functions which is more complex to apply. The product formula method following Peskin [12,13] has the advantage that it can achieve any desired accuracy by using one dimensional discrete delta functions with corresponding discrete moment conditions, as proved in [18]. However the high order version of the product formula method has not been implemented when the curve Γ is represented by a level set function. In [15] Smereka proposed both a first and second order accurate approximate delta function by using a technique for solving elliptic equations with discontinuous source terms. The proof of accuracy of this approximate delta function is presented in [1]. In [19] Towers proposed both a first and second order accurate approximate delta function by using difference approximations to derivatives of the smoothed heaviside function or those of the integral of the heaviside function. The analysis of accuracy of these methods are considered in [20]. However approximate delta functions higher than second order accuracy for the numerical quadratures (1.3) have not been designed and implemented yet. Therefore to design numerical quadratures (1.3) higher than second order accuracy to the delta function integrals(1.1) remains to be studied.

In this paper we design and analyze a class of high order numerical methods to the delta function integrals (1.1) in two dimension. The strategy of the methods in this paper is different from the numerical quadrature approach. These methods are constructed by considering the approximation of the restriction of the two dimensional delta function integral in each mesh cell. This is also a natural strategy. By using this strategy, Min and Gibou designed in [8,9] a second order geometric integration method for computing (1.1) via decomposing the zero level set Γ into simplices. Such a strategy has also been applied to another type of delta function integrals

$$\int_{\mathbb{R}^n} \alpha(\mathbf{x}) \prod_{i=1}^{n} \delta(\beta_i(\mathbf{x})) d\mathbf{x},\quad n = 1, 2, 3, \tag{1.4}$$

where the common zero points of the level set functions $\beta_i(\mathbf{x})$ are essentially finite number of points in the space. Second to fourth order numerical methods to (1.4) in one to three dimensions have been designed in [23]. In order to obtain suitable approximation to the restriction of the delta function integral (1.4) in a mesh cell, the methods in [23] involve checking the existence of common zero points of level set functions and applying the technique of changing interpolation space.

Naturally the methods in this paper also involve checking the existence of zero points of the level set function in the delta function integral (1.1). Namely we need to check the intersection between a mesh cell and the zero level set Γ of the level set function. Our strategy to approximate the restriction of the delta function integral (1.1) in a mesh cell intersecting with Γ is based on the fact that the two dimensional delta function integral in the mesh cell can be rewritten as a one dimensional ordinary integral with the smooth integrand being a one dimensional delta function integral. The transformed one dimensional integral takes one of two forms according to the comparison of the two components of gradient of u in the cell which can be checked from the mesh point values of u. Therefore high order numerical methods to approximate the mesh cell restriction of the two dimensional delta function integral (1.1) in principle can be constructed by applying high order numerical quadratures to one dimensional ordinary integrals and high order numerical methods to one dimensional delta function integrals. The high order numerical quadratures to ordinary integrals are standard. The high order numerical methods to one dimensional delta function integrals have already been studied [23,24]. In this paper we apply the high order numerical method designed in [23] to a type of delta function integrals including the one dimensional case. The algorithm so designed to approximate the mesh cell restrictions of the two dimensional delta function integral (1.1) comprises the numerical method proposed in this paper. The method contains several sub-algorithms including Newton iteration to solve one dimensional interpolation polynomials, Numerical quadrature to one dimensional ordinary integrals, difference approximation formula, interpolation formula and numerical method to approximate one dimensional delta function integrals. We carry out error analysis for the method proposed in this paper and prove that the method can achieve any desired accuracy to the two dimensional delta function integrals (1.1) provided the sub-algorithms in the method attain the corresponding accuracy which is straightforward to fulfill. We also prove the better accuracy of the method in this paper under an assumption on