Journal of Computational Physics 228 (2009) 4291–4302

Contents lists available at ScienceDirect

Journal of Computational Physics

journal homepage: www.elsevier.com/locate/jcp

Adiabatic perturbations for compactons under dissipation and numerically-induced dissipation

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ARTICLE INFO

Article history:
Received 18 April 2008
Received in revised form 9 March 2009
Accepted 9 March 2009
Available online 17 March 2009

MSC:
34E10
35Q53
58J37
65M60

Keywords:
Perturbation methods
Adiabatic perturbations
Numerical methods
Compactons
Numerically-induced tails

ABSTRACT

Compacton propagation under dissipation shows amplitude damping and the generation of tails. The numerical simulation of compactons by means of dissipative schemes also show the same behaviors. The truncation error terms of a numerical method can be considered as a perturbation of the original partial differential equation and perturbation methods can be applied to its analysis. For dissipative schemes, or when artificial dissipation is added, the adiabatic perturbation method yields evolution equations for the amplitude loss in the numerical solution and the amplitude of the numerically-induced tails. In this paper, such methods are applied to the $K(2,2)$ Rosenau–Hyman equation, showing a very good agreement between perturbative and numerical results.

1. Introduction

Perturbation or asymptotic methods [1] can be used for the analysis of the errors introduced by numerical methods when the local truncation error is considered as a perturbation of the original differential equation. For the initial value problem in ordinary differential equations, regular and singular perturbation methods have been straightforwardly applied with success [2,3, and references therein]. For nonlinear evolution equations, the application of perturbation methods in such a context is more difficult, being only scarcely presented in the literature. A few exceptions require attention. Herman and Kickerbocker [4] use direct perturbations not based in the inverse scattering transform in order to study the numerically-induced phase shift in solitons of the Korteweg-de Vries equation propagated by means of the Zabusky–Kruskal scheme. Similar results have been obtained by Marchant and Smyth [5] and Marchant [6] for generalizations of, respectively, the Korteweg-de Vries and the Benjamin–Bona–Mahoney equations. Recently, Junk et al. [7] have studied by asymptotic methods the finite discrete-velocity method for the lattice Boltzmann equation, determining order-by-order the accuracy and structure of the error of the numerical equivalents for the flow velocity, pressure, and vorticity.

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0021-9991/$ - see front matter
doi:10.1016/j.jcp.2009.03.005

The solitary wave solutions of generalized Korteweg-de Vries may have compact support, the so-called compactons, instead of presenting exponentially decreasing tails, characteristic of solitons. Let us consider the $K(2,2)$ compacton equation by Rosenau and Hyman [8], given by

$$u_t - c_0 u_x + (u^2)_x + (u^2)_{xxx} = 0, \tag{1}$$

where $u(x,t)$ is the wave amplitude, x is the spatial coordinate, t is time, c_0 is a constant velocity, and the subindex indicate differentiation. The compacton solution of Eq. (1) is given by

$$u_c(x,t) = \frac{4c}{3}\cos^2\left(\frac{x-(c-c_0)t}{4}\right), \quad |x-(c-c_0)t| \leqslant 2\pi, \tag{2}$$

where c is the velocity of the compacton.

Numerical solutions of the $K(2,2)$ equation show several numerically-induced phenomena, such as spurious radiation [9] or "artificial tails" [10]. Perturbation methods may be applied in order to understand these phenomena and to estimate their magnitudes, however, no general perturbation theory for compactons has been developed in the past. Recently, Pikovsky and Rosenau [10] have applied the method of adiabatic perturbations to compactons. This method is widely known in soliton theory [11–14], yielding the evolution of the soliton parameters on a slow time variable resulting from that of the invariants of the partial differential equation. This method is applicable only for dissipative perturbations. In Ref. [10], only second- and fourth-order linear dissipation for compactons of the $K(2,2)$ equation have been studied.

In this paper, the adiabatic perturbation method is applied to the analysis of the numerically-induced phenomena in the numerical integration of Eq. (1) by means of two numerical methods, based on either the implicit Euler or the implicit midpoint rule in time with a fourth-order spatial discretization, with and without "artificial" dissipation. Sections 2 and 2.1 present both numerical methods and representative numerical results illustrating the damping of the numerical compactons and the generation of tails. Section 3, after briefly reviewing the adiabatic perturbation method for Eq. (1), presents its application to the implicit Euler method in Section 3.1 without "artificial" dissipation and in Section 3.2 for both methods with "artificial" dissipation. In Section 4 the perturbative results are compared with those obtained by the numerical methods in order to determine their scope of validity. Section 5 is devoted to the main conclusions and future lines of research. Finally, an Appendix A detailing the derivation of some equations is included.

2. Numerical methods

Let us consider the numerical solution of the compacton Eq. (1) by means of a Petrov–Galerkin approximation in space with periodic boundary conditions, using C^0 continuous piecewise linear interpolants as trial functions and C^2 continuous Schoenberg cubic B-splines test functions. For the nonlinear terms, the product approximation is applied. The resulting weak formulation for Eq. (1) is as follows: Find a function

$$u(x,t) = \sum_{j=0}^{N} U_j(t)\phi_j(x),$$

such that

$$\langle U_t, \psi_k\rangle - c_0\langle U_x, \psi_k\rangle + \langle (U^2)_x, \psi_k\rangle + \langle (U^2)_x, (\psi_k)_{xx}\rangle = 0, \tag{3}$$

for all $\psi_k(x), k = 0, 1, \ldots, N$, where a uniform mesh is used, $x_j = x_0 + j\Delta x$, the inner product is

$$\langle f, g\rangle = \int_{x_0}^{x_N} f(x)g(x)dx,$$

$U_j(t) = U(x_j, t)$ approximates $u(x_j, t)$, $\phi_j(x)$ are the usual piecewise linear hat functions associated with the node x_j ($\phi_j(x_k) = \delta_{jk}$, the Kronecker delta), and $\psi_k(x)$ are cubic B-splines defined in a $4\Delta x$ interval, which are C^2 continuous as required by Eq. (3).

The evaluation of the inner products in Eq. (3), applying the product approximation, yields the following system of ordinary differential equations

$$\mathcal{A}(\mathrm{E})\frac{dU_j}{dt} - c_0\mathcal{B}(\mathrm{E})U_j + \mathcal{B}(\mathrm{E})(U_j)^2 + \mathcal{C}(\mathrm{E})(U_j)^2 = 0, \tag{4}$$

where E is the shift operator, i.e., $\mathrm{E}U_j = U_{j+1}$ and

$$\mathcal{A}(\mathrm{E}) = \frac{\mathrm{E}^{-2} + 26\mathrm{E}^{-1} + 66 + 26\mathrm{E}^{1} + \mathrm{E}^{2}}{120},$$

$$\mathcal{B}(\mathrm{E}) = \frac{-\mathrm{E}^{-2} - 10\mathrm{E}^{-1} + 10\mathrm{E}^{1} + \mathrm{E}^{2}}{24\Delta x},$$

$$\mathcal{C}(\mathrm{E}) = \frac{-\mathrm{E}^{-2} + 2\mathrm{E}^{-1} - 2\mathrm{E}^{1} + \mathrm{E}^{2}}{2\Delta x^3}.$$