Available online at www.sciencedirect.com
SCIENCE DIRECT®
Journal of Computational Physics 216 (2006) 687–706
JOURNAL OF COMPUTATIONAL PHYSICS
www.elsevier.com/locate/jcp

Wiener Chaos expansions and numerical solutions of randomly forced equations of fluid mechanics

Thomas Y. Hou *, Wuan Luo, Boris Rozovskii, Hao-Min Zhou

Department of Applied Mathematics, California Institute of Technology, 217-50, Pasadena, CA 91125, United States

Received 7 November 2004; received in revised form 11 November 2005; accepted 3 January 2006
Available online 17 February 2006

Abstract

In this paper, we propose a numerical method based on Wiener Chaos expansion and apply it to solve the stochastic Burgers and Navier–Stokes equations driven by Brownian motion. The main advantage of the Wiener Chaos approach is that it allows for the separation of random and deterministic effects in a rigorous and effective manner. The separation principle effectively reduces a stochastic equation to its associated *propagator*, a system of deterministic equations for the coefficients of the Wiener Chaos expansion. Simple formulas for statistical moments of the stochastic solution are presented. These formulas only involve the solutions of the propagator. We demonstrate that for short time solutions the numerical methods based on the Wiener Chaos expansion are more efficient and accurate than those based on the Monte Carlo simulations.

Keywords: Wiener Chaos expansions; Stochastic differential equations; Numerical methods; Navier–Stokes equations

1. Introduction

Stochastic partial differential equations (SPDEs) are known to be an effective tool in modeling complicated phenomena. Examples include wave propagation [25], diffusion through heterogeneous random media [26], randomly forced Burgers and Navier–Stokes equations (see e.g. [2,5,15,23] and the references therein). Additional examples can be found in material science, chemistry, biology, and other areas. Many of the small scale effects and various uncertainties in these problems, which are difficult to deal with using traditional methods, can be naturally modeled by stochastic processes. Unlike deterministic partial differential equations, solutions of SPDEs are random fields. Hence, it is important to be able to deal with their statistical characteristics, e.g. mean, variance, and higher-order moments.

Due to the relatively complex nature of SPDEs, numerical simulations play an important role in the exploration of this important and useful class of PDEs. Currently, the Monte Carlo method is by far the most popular in simulating the effects modeled by SPDEs. The Monte Carlo method and its modifications, however,

* Corresponding author. Tel.: +1 626 395 4546; fax: +1 626 578 0124.
E-mail address: hou@ama.caltech.edu (T.Y. Hou).

0021-9991/$ - see front matter
doi:10.1016/j.jcp.2006.01.008

have some limitations in applications to SPDEs with complex stochastic forcing, e.g., PDEs driven by Brownian motion. To correctly simulate small scale effects, one has to use a fine mesh to resolve the smallest possible scales. Moreover, many realizations have to be computed in order to obtain reliable estimates of various statistical properties. Therefore, Monte Carlo simulations are generally very expensive.

In this paper, we deal with Burgers and Navier–Stokes equation with Brownian-motion type forcing. These equations are important due to their role in modeling of certain turbulent effects (for detailed discussions, see [1,15,17] and the references therein).

The method of attack is based on the Fourier analysis with respect to the Gaussian (rather than Lebesgue) measure. More specifically, the approach taken in this paper is based on the Cameron–Martin version of the Wiener Chaos expansion (see, e.g. [3,12] and the references therein). Sometimes, the Wiener Chaos expansion (WCE for short) is also referred to as the Hermite polynomial chaos expansion.

The WCE of a solution of an SPDE separates the deterministic effects from the randomness. In particular, the Fourier coefficients of a white noise driven Navier–Stokes equation verify a system of coupled deterministic PDEs, which we refer to as the *propagator* (see, e.g. [22,23]). The propagator is a deterministic mechanism responsible for the evolution of randomness inherent to the original stochastic PDE. Remarkably, the propagator has the same type of nonlinearity as the original equation. Once the propagator is determined, standard deterministic numerical methods can be applied to solve it efficiently. The main statistics (such as mean, covariance, and higher-order statistical moments) can be calculated by simple formulas involving only the coefficients of the propagator. In the WCE approach, there is no randomness directly involved in the computations. One does not have to rely upon pseudo-random number generators, and there is no need to solve the stochastic PDEs repeatedly over many realizations. Instead, the propagator system is solved only once.

There is a long history of using WCE as well as other polynomial chaos expansions in problems in physics and engineering. See, e.g. [9,24,6,7], etc. In particular, the papers [9,24,6,7] deal with the contribution of low-order Wiener Chaos approximations to the inertial range spectrum of the Burgers equation. Ghanem and other authors have developed stochastic finite element methods based on the Karhunen–Loève expansion and Hermite polynomial chaos expansion [11,27]. Karniadakis et al. generalized this idea to other types of randomness and polynomials [13,30,31].

As mentioned above, this paper is mainly concerned with the equations of fluid dynamics driven by Brownian motion. Typically, applications of the polynomial chaos methodology to stochastic PDEs considered in the literature deal with stochastic input generated by a finite number of random variables (see, e.g. [27,11,30,29]). Usually, this assumption is introduced either directly or via a representation of the stochastic input by a *truncated* Karhunen–Loève expansion. In contrast, in our case it is necessary to deal constantly with a flux of new random variables (successive time increments of the driving Brownian motions). This effect complicates the problem drastically, even on a short time interval.

To mitigate this problem, we have introduced a compression technique that allows to reduce the number of Hermite polynomials in the expansion quite dramatically. The idea of the method is similar to the sparse tensor products developed by Schwab, etc. [10]. Also, to enhance the accuracy of the numerical simulations, we take advantage of the recently discovered analytical formulae for the coefficients of the WCE (see [23]). Previously, in the literature on numerical analysis of SPDEs, the coefficients were modeled numerically (Ghanem, personal communication).

The statistical characteristics of fluid flow forced by Brownian motion is far from being Gaussian (see, e.g. [7,24]). To address this problem, we have implemented a WCE based technique for computing moments of all orders. It is based on rigorous analytical formulae that involve only the deterministic coefficients of the WCE. In the past, numerical approximations based on polynomial chaos approach to equation of fluid dynamics dealt only with the first- and the second-order moments. In our simulations the statistical moments up to the fourth order were computed.

Practical application of the WCE requires a two way truncation: (a) with respect to the number K of the random variables/wavelets and (b) with respect to the order N of the Hermite polynomials involved. We have addressed the related error analysis problems both analytically and numerically. In particular, it was established that, at least in the case of spatially invariant forcing, the truncation error at time Δt is given by the (asymptotic) formula