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On level set regularization for highly ill-posed distributed parameter estimation problems

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Abstract

The recovery of a distributed parameter function with discontinuities from inverse problems with elliptic forward PDEs is fraught with theoretical and practical difficulties. Better results are obtained for problems where the solution may take on at each point only one of two values, thus yielding a shape recovery problem.

This article considers level set regularization for such problems. However, rather than explicitly integrating a time embedded PDE to steady state, which typically requires thousands of iterations, methods based on Gauss–Newton are applied more directly. One of these can be viewed as damped Gauss–Newton utilized to approximate the steady state equations which in turn are viewed as the necessary conditions of a Tikhonov-type regularization with a sharpening sub-step at each iteration. In practice this method is eclipsed, however, by a special "finite time" or Levenberg–Marquardt-type method which we call *dynamic regularization* applied to the output least squares formulation. Our stopping criterion for the iteration does not involve knowledge of the true solution.

The regularization functional is applied to the (smooth) level set function rather than to the discontinuous function to be recovered, and the second focus of this article is on selecting this functional. Typical choices may lead to flat level sets that in turn cause ill-conditioning. We propose a new, quartic, non-local regularization term that penalizes flatness and produces a smooth level set evolution, and compare its performance to more usual choices.

Two numerical test cases are considered: a potential problem and the classical EIT/DC resistivity problem.

Keywords: Inverse problem; Dynamic regularization; Level set; Gauss–Newton; Levenberg–Marquardt; Electrical impedance tomography; Potential; DC resistivity

1. Introduction

Consider the following data inversion problem. A *forward operator*, $F(m)$, is given, and a *model* $m(\mathbf{x})$ is sought over a discretized domain Ω in 2D or 3D, such that $F(m)$ matches given data b up to the noise level in the data measurements. The forward model is further given by

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$$F(m) = Qu, \tag{1a}$$
$$u = G(m), \tag{1b}$$

where Q is a matrix which projects to data locations (e.g., along the boundary $\partial\Omega$), and $G(m)$ is the inverse of an elliptic PDE system discretized on a grid at least as fine as that of m using a finite volume or finite element method.

Several applications give rise to such a problem formulation. These include DC resistivity [35], linear potential problems [23,7], magnetotelluric inversion [30], diffraction tomography [12], oil reservoir simulation [14] aquifer calibration [17], electrical impedance tomography (EIT) [5,10,6], and Maxwell's equations in low frequencies [26,27,20,21].

It is well-known that while the forward problem (1) is well-posed the inverse problem is not. Indeed, the applications mentioned above and those considered in this article are *highly* ill-posed. In practice for the available noisy data typically there is no unique solution, i.e., there are many models m which yield a field u such that Qu is close to b to within the noise level, and moreover, such models m may vary wildly and depend discontinuously on the data. A direct application of the *output least squares method*, which is to solve the optimization problem

$$\min_m \phi_0 = \frac{1}{2}\|F(m) - b\|^2 \tag{2}$$

using the least squares norm of the data fitting term, typically runs into trouble. In a Tikhonov-type regularization [36], therefore, one approximately solves the optimization problem

$$\min_m \phi_\beta = \frac{1}{2}\|F(m) - b\|^2 + \beta R(m), \tag{3}$$

where $R(m)$ is a regularization term, and $\beta > 0$ is the regularization parameter whose choice has been the subject of many papers (see, e.g., [37]).

For the regularization term, we have considered in previous articles a same-grid discretization of

$$R(m) = \int_\Omega (\rho(|\nabla m|) + \hat{\alpha}(m - m_{\mathrm{ref}})^2)\,\mathrm{d}\mathbf{x}, \tag{4}$$

where $\hat{\alpha}$ is a (typically very small, positive) parameter and m_{ref} is a given reference function (typically, the half-space solution). A least squares regularization is achieved by choosing $\rho(\tau) = \frac{1}{2}\tau^2$, although we have used also a weighted least squares penalty function, see [19,1,2,21,4].

However, the least squares functional is well-known to be unsuitable if a priori information that the model m contains *discontinuities* is to be respected. Total variation regularization has been proposed and successfully applied to denoising and mildly ill-posed problems such as deblurring [31]. In [4,3], we have discussed and developed further the use of modified total variation (TV), or (occasionally slightly better) Huber switching between TV and least squares. But we also demonstrated that these methods may fail when applied to highly ill-posed problems such as those considered in this article. Specifically, examples showed that simply trusting the reconstruction because the data misfit is small cannot be advocated, even if the discrepancy principle is obeyed and even under the unrealistic assumption that data is available everywhere. It can then be argued that displaying a smooth blob, such as when using weighted least squares regularization, is less committing and more truthful to the quality of the actual information at hand than displaying a discontinuous solution. For recovering sharp discontinuous solutions more a priori information is required.

Such additional information is available if we know that the model function $m(\mathbf{x})$ is piecewise constant. In fact, let us assume in this article that m may only take on one of two known values, m_1 and m_2 (say, a homogeneous body and a homogeneous background). The problem becomes that of shape optimization, and a level set approach [34,29,8] for incorporating this additional information is natural.

Following [32,24,18] and others we consider $m(\mathbf{x})$ as a function of a smoother one $\psi(\mathbf{x})$ and apply regularization to ψ,