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Short Note

ADI-SGS scheme on ideal magnetohydrodynamics

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1. Introduction

The magnetohydrodynamics (MHD) is often used to describe many important problems in astrophysics, space propulsions, magnetic-confinement fusion and so on. For these problems, the numerical simulation is a powerful and important analytical tool.

Time-marching MHD simulations are, in particular, challenging because the MHD model contains a wide range of time scales even in the ideal MHD limit. In the ideal MHD model, there are three types of MHD waves, each with a characteristic wave speed; the fast wave, the Alfvén wave and the slow wave. Although the transit time of the MHD wave is very small in the presence of a strong magnetic field, research often focuses on the steady states of the field and phenomena with much a longer time scale. Thus, for explicit time integration, the calculations need an enormous number of time steps because the time step interval is restricted by the fastest MHD wave (the fast wave) from the CFL conditions.

There have been a number of applications of the implicit methods to MHD equations (for example, see [1–5]). Whereas standard central difference methods were used in most previous works, few have applied an approximate Riemann solver to implicit methods [1,2].

The motivation of our study is to introduce a simple and easy to code implicit scheme to MHD, using an approximate Riemann solver and a Jacobian-free technique. We adopted a fully implicit scheme called the ADI-SGS scheme [6,7], which is used for hydrodynamic equations, and simply applied this scheme to ideal one-fluid MHD equations. The ADI-SGS scheme is derived by combining alternative direction implicit (ADI) factorization [8] with the lower–upper symmetric–Gauss–Seidel (LU-SGS) method [9]. Coding of the ADI-SGS scheme is easier than the ADI scheme, and vectorization or parallelization is much easier than the LU-SGS scheme. Therefore, the ADI-SGS scheme seems to be more suitable for large scale computing.

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2. Formulation of the ADI-SGS scheme

The basic formulations are briefly explained. The ADI-SGS scheme can be derived by combining ADI factorization and the LU-SGS method. Detailed derivations of the formulations can be found in the references noted in this section.

The ideal MHD equations are expressed in general curvilinear coordinates (ξ, η, ζ) as follows [10]:

$$\frac{\partial \boldsymbol{Q}}{\partial t} = -\left(\frac{\partial \boldsymbol{E}}{\partial \xi} + \frac{\partial \boldsymbol{F}}{\partial \eta} + \frac{\partial \boldsymbol{G}}{\partial \zeta}\right). \tag{1}$$

Here, $\boldsymbol{Q}$ is the vector of conservative variables, and $\boldsymbol{E}, \boldsymbol{F}$ and $\boldsymbol{G}$ are the flux vectors in the ξ, η and ζ directions, respectively. The detail general curvilinear coordinate transformations of the MHD equations are in Ref. [10]. In an implicit scheme, the right-hand side of Eq. (1) is evaluated at the $n+1$ time level, and Eq. (1) can be written by discretizing the time term in the first-order form as

$$\Delta \boldsymbol{Q}^n = -\Delta t\left(\frac{\partial \boldsymbol{E}}{\partial \xi} + \frac{\partial \boldsymbol{F}}{\partial \eta} + \frac{\partial \boldsymbol{G}}{\partial \zeta}\right)^{n+1}. \tag{2}$$

Here, Δt is the time step interval and $\Delta \boldsymbol{Q}^n = \boldsymbol{Q}^{n+1} - \boldsymbol{Q}^n$. Eq. (2) is rewritten using the linearization of the flux vectors proposed by Beam and Warming [8], and then is expressed in the grid point as follows:

$$\left(\boldsymbol{I} + \Delta t\frac{\partial}{\partial \xi}\boldsymbol{A}^n + \Delta t\frac{\partial}{\partial \eta}\boldsymbol{B}^n + \Delta t\frac{\partial}{\partial \zeta}\boldsymbol{C}^n\right)_{i,j,k}\Delta \boldsymbol{Q}^n_{i,j,k} = -\Delta t\left(\frac{\partial \boldsymbol{E}}{\partial \xi} + \frac{\partial \boldsymbol{F}}{\partial \eta} + \frac{\partial \boldsymbol{G}}{\partial \zeta}\right)^n_{i,j,k}. \tag{3}$$

Here, $\boldsymbol{A}, \boldsymbol{B}$ and $\boldsymbol{C}$ are the flux Jacobian matrix of $\boldsymbol{E}, \boldsymbol{F}$ and $\boldsymbol{G}$, respectively. In a finite-volume method, $\boldsymbol{Q}_{i,j,k}$ is the cell-averaged conserved variable vector, and the numerical fluxes on the right side of the Eq. (3) are evaluated at the cell interface by any MHD solvers. Eq. (3) may be modified specifically by the ADI factorization proposed by Beam and Warming [8] as described below:

$$\left(\boldsymbol{I} + \Delta t\frac{\partial}{\partial \xi}\boldsymbol{A}^n\right)_{i,j,k}\left(\boldsymbol{I} + \Delta t\frac{\partial}{\partial \eta}\boldsymbol{B}^n\right)_{i,j,k}\left(\boldsymbol{I} + \Delta t\frac{\partial}{\partial \zeta}\boldsymbol{C}^n\right)_{i,j,k}\Delta \boldsymbol{Q}^n_{i,j,k} = \boldsymbol{RHS}^n_{i,j,k}. \tag{4}$$

Three operators in the left-hand side of this equation are calculated in order. We adopt the LU-SGS method [9] to calculate each operator.

Diagonally dominant LDU factorization is applied to each directional operator on the left side of Eq. (4). This factorization is used in LU-ADI scheme [11] and LU-SGS scheme [9]. The operator in the ξ direction becomes

$$\left(\boldsymbol{I} - \frac{\Delta t}{\Delta \xi}\boldsymbol{A}^{n-}_{i,j,k} + \Delta t\delta^b_\xi \boldsymbol{A}^{n+}\right)\left(\boldsymbol{I} + \frac{\Delta t}{\Delta \xi}\left(\boldsymbol{A}^{n+}_{i,j,k} - \boldsymbol{A}^{n-}_{i,j,k}\right)\right)^{-1}\left(\boldsymbol{I} + \frac{\Delta t}{\Delta \xi}\boldsymbol{A}^{n+}_{i,j,k} + \Delta t\delta^f_\xi \boldsymbol{A}^{n-}\right)\Delta \tilde{\tilde{\boldsymbol{Q}}}^n_{i,j,k} = \boldsymbol{RHS}^n_{i,j,k}. \tag{5}$$

Here, $\boldsymbol{A}^+$ and $\boldsymbol{A}^-$ are the flux Jacobian matrices which have only positive and negative eigenvalues, and δ^f_ξ and δ^b_ξ are the forward and backward derivatives, respectively. $\Delta \tilde{\tilde{\boldsymbol{Q}}}^n_{i,j,k}$ is

$$\Delta \tilde{\tilde{\boldsymbol{Q}}}^n_{i,j,k} = \left(\boldsymbol{I} + \Delta t\frac{\partial}{\partial \eta}\boldsymbol{B}^n\right)_{i,j,k}\left(\boldsymbol{I} + \Delta t\frac{\partial}{\partial \zeta}\boldsymbol{C}^n\right)_{i,j,k}\Delta \boldsymbol{Q}^n_{i,j,k}. \tag{6}$$

The operators of Eq. (5), $(\boldsymbol{I} - \Delta t\boldsymbol{A}^{n-}_{i,j,k}/\Delta\xi + \Delta t\delta^b_\xi \boldsymbol{A}^{n+})$ and $(\boldsymbol{I} + \Delta t\boldsymbol{A}^{n+}_{i,j,k}/\Delta\xi + \Delta t\delta^f_\xi \boldsymbol{A}^{n-})$, lead to the lower (L) and upper (U) block-triangular matrices, respectively, and $(\boldsymbol{I} + \Delta t(\boldsymbol{A}^{n+}_{i,j,k} - \boldsymbol{A}^{n-}_{i,j,k})/\Delta\xi)^{-1}$ leads to the block-diagonal (D) matrix. Eq. (5) is calculated in two steps as follows:

1. $$\left(\boldsymbol{I} - \frac{\Delta t}{\Delta \xi}\boldsymbol{A}^{n-}_{i,j,k} + \Delta t\delta^b_\xi \boldsymbol{A}^{n+}\right)\Delta \tilde{\tilde{\boldsymbol{Q}}}^{n*}_{i,j,k} = \boldsymbol{RHS}^n_{i,j,k}, \tag{7}$$
2. $$\left(\boldsymbol{I} + \frac{\Delta t}{\Delta \xi}\boldsymbol{A}^{n+}_{i,j,k} + \Delta t\delta^f_\xi \boldsymbol{A}^{n-}\right)\Delta \tilde{\tilde{\boldsymbol{Q}}}^n_{i,j,k} = \left(\boldsymbol{I} + \frac{\Delta t}{\Delta \xi}\left(\boldsymbol{A}^{n+}_{i,j,k} - \boldsymbol{A}^{n-}_{i,j,k}\right)\right)\Delta \tilde{\tilde{\boldsymbol{Q}}}^{n*}_{i,j,k}. \tag{8}$$

In the LU-SGS method, $\boldsymbol{A}^\pm$ is approximated as $\boldsymbol{A}^\pm = (\boldsymbol{A} \pm \sigma_\xi)/2$, where σ_ξ is the spectral radius of $\boldsymbol{A}$. From this approximation, we can get the following discretized forms of Eqs. (7) and (8);

1. $$\left(1 + \frac{\Delta t}{\Delta \xi}\sigma^n_{\xi i,j,k}\right)\Delta \tilde{\tilde{\boldsymbol{Q}}}^{n*}_{i,j,k} = \boldsymbol{RHS}^n_{i,j,k} + \frac{\Delta t}{\Delta \xi}\boldsymbol{A}^{n+}_{i-1,j,k}\Delta \tilde{\tilde{\boldsymbol{Q}}}^{n*}_{i-1,j,k}, \tag{9}$$
2. $$\left(1 + \frac{\Delta t}{\Delta \xi}\sigma^n_{\xi i,j,k}\right)\Delta \tilde{\tilde{\boldsymbol{Q}}}^n_{i,j,k} = \left(1 + \frac{\Delta t}{\Delta \xi}\sigma^n_{\xi i,j,k}\right)\Delta \tilde{\tilde{\boldsymbol{Q}}}^n_{i,j,k} - \frac{\Delta t}{\Delta \xi}\boldsymbol{A}^{n-}_{i+1,j,k}\Delta \tilde{\tilde{\boldsymbol{Q}}}^n_{i+1,j,k}. \tag{10}$$

$\Delta \tilde{\tilde{\boldsymbol{Q}}}^{n*}$ can be obtained from the forward i-sweep calculation using Eq. (9), and $\Delta \tilde{\tilde{\boldsymbol{Q}}}^n$ can be obtained from the backward i-sweep calculation using Eq. (10). The operator in the η and ζ directions of Eq. (4) can be calculated using the same procedures, and finally we obtain $\Delta \boldsymbol{Q}^n$.